THE L^P-OPERATOR NORM OF A QUADRATIC PERTURBATION OF THE REAL PART OF THE AHLFORS–BEURLING OPERATOR

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ABSTRACT

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Given a sequence of martingale differences, Burkholder found the sharp constant for the L^p-norm of the corresponding martingale transform. We are able to determine the sharp L^p-norm of small "quadratic perturbations" of the martingale transform in L^p. By "quadratic perturbation" of the martingale transform we mean the L^p norm of the square root of the squares of the martingale transform and the original martingale (with small constant). The problem of perturbation of martingale transform appears naturally if one wants to estimate the linear combination of Riesz transforms (as, for example, in the case of Ahlfors–Beurling operator). Let $\{d_k\}_{k\geq 0}$ be a complex martingale difference in $L^p[0,1]$, where $1<p<\infty$, and $\{\varepsilon_k\}_{k\geq 0}$ a sequence of signs. We obtain the following generalization of Burkholder's famous result. If $n\in\mathbb{Z}_+$, then we have the following estimate

$$\Big\|\sum_{k=0}^{n}(\varepsilon_k,\,\tau)d_k\Big\|_{L^p([0,1],\mathbb{C}^2)}\leq\big((p^*-1)^2+\tau^2\big)^{\frac{1}{2}}\Big\|\sum_{k=0}^{n}d_k\Big\|_{L^p([0,1],\mathbb{C})},$$

with $\big((p^*-1)^2+\tau^2\big)^{\frac{1}{2}}$ being the sharp constant in the estimate for $1<p<2$ and $\tau^2\leq\frac{1}{2p-1}$ or $2\leq p<\infty$ and $\tau\in\mathbb{R}$, where $p^*-1=\max\{p-1,\frac{1}{p-1}\}$. This result is significant not only because it is a generalization of Burkholder's famous result from the 1980's, but also because we can apply to obtaining the exact operator norm of a certain singular integral operator.

Let R_1 and R_2 be the planar Riesz transforms. We compute the L^p-operator norm of a quadratic perturbation of $R_1^2 - R_2^2$ as

$$\left\|(R_1^2 - R_2^2\,, \tau I)\right\|_{L^p(\mathbb{C},\mathbb{C}) \to L^p(\mathbb{C},\mathbb{C}^2)} = \left((p^* - 1)^2 + \tau^2\right)^{\frac{1}{2}},$$

for $1 < p < 2$ and $\tau^2 \leq \frac{1}{2p-1}$ or $2 \leq p < \infty$ and $\tau \in \mathbb{R}$. To obtain the lower bound estimate of, what we are calling a quadratic perturbation of $R_1^2 - R_2^2$, we discuss a new approach of constructing laminates (a special type of probability measure on matrices) to approximate the Riesz transforms.

To my wife Lindsey and daughter Elizabeth

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Chapter 1

Introduction

Determining the exact L^p-operator norm of a singular integral operator is a difficult task to accomplish in general. Classically, the Hilbert transform's operator norm was determined by Pichorides [32]. More recently, the real and imaginary parts of the Ahlfors–Beurling operator were determined by Nazarov,Volberg [31] and Geiss, Montgomery-Smith, Saksman [24]. Considering the full Alhfors–Beurling operator, the lower bound was determined as $p^* - 1$ by Lehto [28], or a new proof of this fact in [10]. On the other hand the upper bound has been quite a bit more difficult. Iwaniec conjectured in [25] that the upper bound is $p^* - 1$. However, attempts at getting the conjectured upper bound of $p^* - 1$ have been unsuccessful so far. The works of Bañuelos,Wang [4], Nazarov, Volberg [31], Bañuelos, Méndez-Hernández [2], Dragičevič, Volberg [22], Bañuelos, Janakiraman [3], and Borichev, Janakiraman, Volberg [5] have progressively gotten closer to $p^* - 1$ as the upper bound, but no one has yet achieved it. We note that all of these upper bound estimates crucially rely on Burkholder's estimates [15] of the martingale transform. Burkholder's estimates of the martingale transform even play a crucial part in determining the sharp lower bounds of the real and imaginary parts of the Alhfors-Beurling operator as seen in Geiss, Montgomery-Smith, Saksman [24]. Also, in [24] it becomes clear that if one can determine the L^p-operator norm of some perturbation of the martingale transform then one can use it to determine the L^p-operator norm of the same perturbation of the real or imaginary part of the Alhfors-Beurling operator. The focus of Chapter 2 is determining the L^p-operator norm for a “quadratic perturbation” of the martingale transform using the Bellman function technique, which is similar to how Burkholder originally did, for $\tau = 0$, in [15]. By “quadratic perturbation”, we are referring to the quantity $(Y^2 + \tau^2 X^2)^{\frac{1}{2}}$, where $\tau \in \mathbb{R}$ is small, X is a martingale and Y is the corresponding martingale transform.

We claim that our operator

$$\left\|(R_1^2 - R_2^2, \tau I)\right\|_{L^p(\mathbb{C},\mathbb{C}) \to L^p(\mathbb{C},\mathbb{C}^2)},$$

represents a simpler model of the difficulties encountered by treating the Ahlfors–Beurling transform. An extra interesting feature of this operator is that there it breaks the symmetry between $p \in (1, 2)$ and $p \in (2, \infty)$. Another interesting feature is that it is *simpler* to treat than a seemingly simpler perturbation $R_1^2 - R_2^2 + \tau I$, whose norm in L^p is horrendously difficult to find. The last assertion deserves a small elaboration. Burkholder found the norm in L^p of the martingale transform, where the family of transforming multipliers ϵ_I run over $[-1, 1]$. It is not known how (and it seems very difficult) to find the norm of the martingale transform, where the family of transforming multipliers ϵ_I run over $[-0.9, 1.1]$. That would be the norm in L^p of $R_1^2 - R_2^2 + 0.1\, I = 1.1R_1^2 - 0.9R_2^2$. However, if one writes the perturbation in the form we did this above, the problem immediately becomes much more treatable, and we manage to find the precise formula for the norm for a wide range of

p's and τ's.

The method of Bourgain [11], for the Hilbert transform, which was later generalized for a large class of Fourier multiplier operators by Geiss, Montgomery-Smith, Saksman [24], is to discretize the operator and generalize it to a higher dimensional setting. This operator in the higher dimensional setting will turn out to have the same operator norm and it naturally connects with discrete martingales, if done in a careful and clever way. At the end, one has the operator norm of the singular integral bounded below by the operator norm of the martingale transform, which Burkholder found in [15]. This approach can be used for estimating $\left\|(R_1^2 - R_2^2, \tau I)\right\|_{L^p \to L^p}$ from below as well, see [10]. However, we will present an entirely different approach to the problem.

Rather than working with estimates on the martingale transform, we only need to consider the "Burkholder" functions that were used to find those sharp estimates on the martingale transform. More specifically, we analyze the behavior of the Burkholder functions, U and v found in Chapter 2, associated with determining the L^p-operator norm of the quadratic perturbation of the martingale transform. Using the fact that U is the least bi-concave majorant of v (in the appropriately chosen coordinates), in addition to some of the ways in which the two functions interact will allow us to construct an appropriate sequence of laminates, which approximate the push forward of the 2-dimensional Lebesgue measure by the Hessian of a smooth function with compact support. Once the appropriate sequence of laminates is constructed, we are finished since the norm of the Riesz transforms can be approximated by certain fractions of the partial derivatives of smooth functions. The beauty of this method is that it quickly gets us the sharp lower bound constant with very easy calculations. This lower bound argument is discussed in Section 3.2.

The use of laminates for obtaining lower bounds on L^p-estimates has been first recognized by D. Faraco. In [23] he introduced the so-called staircase laminates. These have also been used in refined versions of convex integration in [1] in order to construct special quasiregular mappings with extremal integrability properties. Staircase laminates also proved useful in several other problems, see for instance [21].

In this text we will use a continuous, rather than a discrete laminate. More importantly, in contrast to the techniques used in [23, 1, 21] we will construct the laminate indirectly using duality. The advantage is essentially computational, we very quickly obtain the sharp lower bounds. Indeed, the lower bound follows from the following inequality

$$f(1,1) \leq \frac{1}{(1-k)} \int_1^\infty [f(kt,t) + f(t,kt)] t^{-p_k} \frac{dt}{t},$$

where $k \in (-1,1)$ and $p_k = \frac{2}{1-k}$, and valid for all biconvex functions $f : \mathbb{R}^2 \to \mathbb{R}$ with $f(x) = o(|x|^{p_k})$ as $|x| \to \infty$ (see Section 3.2.3).

The "Burkholder" functions U and v also play a crucial role in obtaining the sharp upper bound estimate as well. With the Burkholder functions we are able to extend sharp estimates of $(Y^2 + \tau^2 X^2)^{\frac{1}{2}}$, obtained in Chapter 2, from the discrete martingale setting to the continuous martingale setting. The use of "heat martingales", as in [2] and [3], will allow us to connect the Riesz transforms to the continuous martingales estimate, without picking up any additional constants. This upper bound argument is presented in Section 3.4.

Chapter 2

A Quadratic Perturbation of the Martingale Transform

2.1 Introduction

In a series of papers, [12] to [19], Burkholder was able to compute the L^p operator norm of the martingale transform, which we will denote as MT. This was quite a revolutionary result, not only because of the result itself but because of the method for approaching the problem. Burkholder's method in these early papers was inspiration for the Bellman function technique, which has been a very useful tool in approaching modern and classical problems in harmonic analysis (this chapter will demonstrate the Bellman function technique as well). But, the result itself has many applications. One particular application of his result is for obtaining sharp estimates for singular integrals. Consider the Ahlfors-Beurling operator, which we will denote as T. Lehto, [28], showed in 1965 that $\|T\|_p := \|T\|_{p \to p} \geq (p^* - 1) = \max\left\{p-1, \frac{1}{p-1}\right\}$. Iwaniec conjectured in 1982, [25], that $\|T\|_p = p^* - 1$. The only progress toward proving that conjecture has been using Burkholder's result, see [31], [2] and [3] for the major results toward proving the conjecture. However, Burkholder's estimates have been useful for lower bound estimates as well. For example, Geiss, Montgomery-Smith and Saksman, [24], were able to show that $\|\Re T\|_p, \|\Im T\|_p \geq p^* - 1$, by using Burkholder's estimates. The upper bound for these two operators were determined as $p^* - 1$ by Nazarov, Volberg, [31] and Bañuelos, Mèndez-Hernàndez [2], so we now have $\|\Re T\|_p = \|\Im T\|_p = p^* - 1$. Note that $\Re T$ the difference of the squares of the planar Riesz transforms, i.e. $T = R_1^2 - R_2^2$.

A recent result of Geiss, Montomery-Smith and Saksman, [24] points to the following observation, though not immediately. We can estimate linear combinations of squares of Riesz transforms if we know the corresponding estimate for a linear combination of the martingale transform and the identity operator. In other words, one can get at estimates of the norm of $(R_1^2 - R_2^2) + \tau \cdot I$, by knowing the estimates of the norm of $MT + \tau \cdot I$. $\|MT + \tau \cdot I\|_p$ has only been computed for either $\tau = 0$ by Burkholder [15] or $\tau = \pm 1$ by Choi [20]. The problem is still open for all other τ-values and seems to be very difficult, though we have had some progress. But, if we consider "quadratic" rather than linear perturbations then things become more manageable. This brings us to the focus of this paper, which is determining estimates for quadratic perturbations of the martingale transform, which will have connections to quadratic combinations of squares of Riesz transforms.

To prove our main result we are going to take a slightly indirect approach. Burkholder (see [15]) defined the martingale transform, MT_ε, as

$$MT_\varepsilon\Big(\sum_{k=1}^{n} d_k\Big) := \sum_{k=1}^{n} \varepsilon_k d_k.$$

Then the main result can be stated as

$$\sup_{\vec{\varepsilon}} \left\| \left(MT_{\vec{\varepsilon}}, \tau I \right) \right\|_{L^p(\mathbb{C}) \to L^p(\mathbb{C}^2)} = \sup_{\vec{\varepsilon}} \frac{\left\| \sum_{k=1}^{n} (\varepsilon_k, \tau) d_k \right\|_p}{\left\| \sum_{k=1}^{n} d_k \right\|_p} = ((p^* - 1)^2 + \tau^2)^{\frac{1}{2}},$$

where I is the identity transformation and τ is "small". However, rather than working with this martingale transform in terms of the martingale differences, in a probabilistic setting, we will define another martingale transform in terms of the Haar expansion of $L^p[0,1]$ functions and set up a Bellman function in that context. Burkholder showed, in [19], that these two different martingale transforms have the same L^p operator norm, for $\tau = 0$, so we expected a perturbation of these to act similarly and it turns out that they do. For convenience, we will work with the martingale transform in the Haar setting. Using the Bellman function technique will turn the problem of finding the sharp constant of the above estimate into solving a second order partial differential equation. The beauty of this approach is that it gets right to the heart of the problem with very little advanced techniques needed in the process. In fact, the only background material that is needed for the Bellman function technique approach, is some basic knowledge of partial differential equations and some elementary analysis.

Observe that for $2 \leq p < \infty$, the estimate from above is just an application of Minkowski's inequality on $L^{\frac{p}{2}}$ and Burkholder's original result. But, this argument does not address sharpness, even though the constant obtained turns out to be the sharp constant for all τ. For $1 < p < 2$, Minkowski's inequality (in $l^{\frac{2}{p}}$) also plays a role, but to a lesser extent and cannot give the sharp constant, as we will see Proposition 33. It is, indeed, very strange that such sloppy estimation could give the estimate with sharp constant for $1 \leq p < \infty$. We will now rigorously develop some background ideas needed to set up the Bellman function.

In our calculations we follow the scheme of [37], but our "Dirichlet problem" for Monge-Ampère is different. For small τ the scheme works. For large τ and $1 < p < 2$ it definitely must be changed, as will be shown later. The amazing feature is the "splitting" of the result to two quite different cases: $1 < p < 2$ and $2 \le p < \infty$, where in the former case we know the result only for small τ, but in the latter one τ is unrestricted.

2.1.1 Motivation of the Bellman function

Let I be an interval and $\alpha^{\pm} \in \mathbb{R}^{+}$ such that $\alpha^{+} + \alpha^{-} = 1$. These $\alpha^{\pm}$ generate two subintervals $I^{\pm}$ such that $|I^{\pm}| = \alpha^{\pm}|I|$ and $I = I^{-} \cup I^{+}$. We can continue this decomposition indefinitely as follows. For any sequence $\{\alpha_{n,m} : 0 < \alpha_{n,m} < 1, 0 \le m < 2^{n}, 0 < n < \infty, \alpha_{n,2k} + \alpha_{n,2k+1} = 1\}$, we can generate the sequence $\mathcal{I} := \{I_{n,m} : 0 \le m < 2^{n}, 0 < n < \infty\}$, where $I_{n,m} = I^{-}_{n,m} \cup I^{+}_{n,m} = I_{n+1,2m+1} \cup I_{n+1,2m+1}$ and $\alpha^{-} = \alpha_{n+1,2m}, \alpha^{+} = \alpha_{n+1,2m+1}$. Note that $I_{0,0} = I$.

For any $J \in \mathcal{I}$ we define the Haar function $h_J := -\sqrt{\frac{\alpha^{+}}{\alpha^{-}|J|}}\chi_{J^{-}} + \sqrt{\frac{\alpha^{-}}{\alpha^{+}|J|}}\chi_{J^{+}}$. If $\max\{|I_{n,m}| : 0 \le m < 2^{n}\} \to 0$ as $n \to \infty$, then $\{h_J\}_{J \in \mathcal{I}}$ is an orthonormal basis for $L^2_0(I) := \{f \in L^2(I) : \int_I f = 0\}$. However, if we add one extra function, then Haar functions form an orthonormal basis in $L^2[0,1]$ function. We will show that this is true in the case $I_0 = [0,1]$ and $\mathcal{I} = \mathcal{D}$ as the dyadic subintervals of I_0. The Bellman function will also be set up under this specific choice of $\mathcal{I} = \mathcal{D}$ for clarity. As it will turn out the Bellman function will be the same under the the choice of $\mathcal{I}$, so the fact that we set up the Bellman function under this specific choice of $\mathcal{I}$ will not matter.

Let $\mathcal{D}_j = \{I \in \mathcal{D} : |I| = 2^{-j}\}$. We use the notation $\langle f \rangle_J$ to represent the average integral of f over the interval $J \in \mathcal{D}$ and $\Delta_j f = \langle f \rangle_{I_{j+1}} - \langle f \rangle_{I_j}$, where $I_j \in \mathcal{D}_j$ and

$I_{j+1} \in \mathcal{D}_{j+1}$. For any $f \in L^1(I_0)$ we have $\Delta_j f = \sum_{I \in \mathcal{D}_j} (f, h_I) h_I$. Then

$$\sum_{j=0}^{\infty} \Delta_j f = \lim_{N \to \infty} \langle f \rangle_{I_{N+1}} - \langle f \rangle_{I_0} \tag{2.1}$$

By Lebesgue differentiation, the limit in (2.1) converges to f almost everywhere as $N \to \infty$. So any $f \in L^p(I_0) \subset L^1(I_0)$ can be decomposed in terms of the Haar system as

$$f = \langle f \rangle_{I_0} \chi_{I_0} + \sum_{I \in \mathcal{D}} (f, h_I) h_I.$$

In terms of the expansion in the Haar system we define the martingale transform, g of f, as

$$g := \langle g \rangle_{I_0} \chi_{I_0} + \sum_{I \in \mathcal{D}} \varepsilon_I (f, h_I) h_I,$$

where $\varepsilon_I \in \{\pm 1\}$. Requiring that $|(g, h_J)| = |(f, h_J)|$, for all $J \in \mathcal{D}$, is equivalent to g being the martingale transform of f, for $f, g \in L^p(I_0)$.

Now we define the Bellman function as $\mathcal{B}(x_1, x_2, x_3) :=$

$$\sup_{f,g} \left\{ \langle (g^2 + \tau^2 f^2)^{\frac{p}{2}} \rangle_I : x_1 = \langle f \rangle_I, x_2 = \langle g \rangle_I, x_3 = \langle |f|^p \rangle_I, |(f, h_J)| = |(g, h_J)|, \forall J \in \mathcal{D} \right\}$$

on the domain $\Omega = \{x \in \mathbb{R}^3 : x_3 \geq 0, |x_1|^p \leq x_3\}$. The Bellman function is defined in this way, since we would like to know the value of the supremum of $\|(g, \tau f)\|_p$, where g is the martingale transform of f. Note that $|x_1|^p \leq x_3$ is just Hölder's inequality. Even though the Bellman function is only being defined for real-valued functions, we can "vectorize" it to work for complex-valued (and even Hilbert-valued) functions, as we will later demonstrate.

Finding the Bellman function will make proving the following main result quite easy. We will call $\langle (g^2+\tau^2 f^2)^{\frac{p}{2}}\rangle_I^{\frac{1}{p}}$ the "quadratic perturbation" of the martingale transform's norm $\langle |g|^p\rangle_I^{\frac{1}{p}}$.

Theorem 1. *Let $\{d_k\}_{k\geq 1}$ be a complex martingale difference in $L^p[0,1]$, where $1<p<\infty$, and $\{\varepsilon_k\}_{k\geq 1}$ a sequence in $\{\pm 1\}$. If $n\in\mathbb{Z}_+$, then*

$$\Big\|\sum_{k=1}^{n}(\varepsilon_k,\,\tau)d_k\Big\|_{L^p([0,1],\mathbb{C}^2)}\leq \big((p^*-1)^2+\tau^2\big)^{\frac{1}{2}}\Big\|\sum_{k=1}^{n}d_k\Big\|_{L^p([0,1],\mathbb{C})},$$

with $\big((p^*-1)^2+\tau^2\big)^{\frac{p}{2}}$ as the sharp constant for $\tau^2\leq\frac{1}{2p-1}$ and $1<p<2$ or $\tau\in\mathbb{R}$ and $2\leq p<\infty$.

Note that when $\tau=0$ we get Burkholder's famous result [15].

Now that we have the problem formalized, notice that $\mathcal{B}$ is independent of the initial choice of I_0 (which we will just denote I from now on) and $\{\alpha_{n,m}\}_{n,m}$, so we return to having them arbitrary. Finding $\mathcal{B}$ when $p=2$ is easy, so we will do this first.

Proposition 2. *If $p=2$ then $\mathcal{B}(x)=x_2^2-x_1^2+(1+\tau^2)x_3$.*

Proof. Since $f\in L^2(I)$ then $f=\langle f\rangle_I\chi_I+\sum_{J\in\mathcal{D}}(f,h_J)h_J$ implies

$$\begin{aligned}\langle |f|^2\rangle_I&=\frac{1}{|I|}\int_I|f|^2\\&=\langle f\rangle_I^2+2\langle f\rangle_I\sum_{J\in\mathcal{D}}(f,h_J)\frac{1}{|I|}\int_I h_J+\frac{1}{|I|}\int_I\sum_{J,K\in\mathcal{D}}(f,h_J)(f,h_K)h_Jh_K\\&=\langle f\rangle_I^2+\frac{1}{|I|}\sum_{J\in\mathcal{D}}|(f,h_J)|^2.\end{aligned}$$

So $\|f\|_2^2 = |I|x_3 = |I|x_1^2 + \sum_{J\in\mathcal{D}} |(f,h_J)|^2$ and similarly

$$\|g\|_2^2 = |I|x_2^2 + \sum_{J\in\mathcal{D}} |(g,h_J)|^2 = |I|x_2^2 + \sum_{J\in\mathcal{D}} |(f,h_J)|^2.$$

Now we can compute $\mathcal{B}$ explicitly, $(p=2)$

$$\langle (g^2+\tau^2 f^2)^{\frac{p}{2}} \rangle_I = \langle |g|^2 \rangle_I + \tau^2 \langle |f|^2 \rangle_I = x_2^2 + \tau^2 x_1^2 + (1+\tau^2)\frac{1}{|I|}\sum_{J\in\mathcal{D}} |(f,h_J)|^2$$
$$= x_2^2 + \tau^2 x_1^2 + (1+\tau^2)(x_3 - x_1^2). \qquad \square$$

Remark 3. The fact that τ remains restricted when p approaches 2 from the left is by-product of a certain inefficiency in the proof. But for the time being we do not know how to lighten the restriction on τ when p approaches 2 from the left.

2.1.2 Outline of Argument to Prove Main Result

Computing the Bellman function, $\mathcal{B}$, for $p \neq 2$, is much more difficult, so more machinery is needed. In Section 2.1.3 we will derive properties of the Bellman function, the most notable of which is concavity under certain conditions. Finding a $\mathcal{B}$ to satisfy the concavity will amount to solving a partial differential equation, after adding an assumption. This PDE has a solution on characteristics that is well known, so we just need to find an explicit solution from this, using the Bellman function properties. How the characteristics behave in the domain of definition for the Bellman function will give us several cases to consider. In Section 2.2 we will get a Bellman function candidate for $1 < p < \infty$ by putting together several cases. Once we have what we think is the Bellman function, we need to show that it has the necessary smoothness and that Assumption 8 was not too restrictive to give us the Bellman function. This is covered in Section 2.3. Finally the main result is shown in Section 2.4. In Section 2.6, we show why several cases did not lead to a Bellman function candidate and why the value of τ was restricted for the Bellman function candidate.

2.1.3 Properties of the Bellman function

One of the properties we nearly always have (or impose) for any Bellman function, is concavity (or convexity). It is not true that $\mathcal{B}$ is globally concave, on all of Ω, but under certain conditions it is concave. The needed condition is that g is the martingale transform of f, or $|x_1^+ - x_1^-| = |x_2^+ - x_2^-|$ in terms of the variables in Ω.

Definition 4. We say that the function B on Ω has restrictive concavity if for all $x^\pm \in \Omega$ such that $x = \alpha^+ x^+ + \alpha^- x^-, \alpha^+ + \alpha^- = 1$ and $|x_1^+ - x_1^-| = |x_2^+ - x_2^-|$, then $\mathcal{B}(x) \geq \alpha^+ \mathcal{B}(x^+) + \alpha^- \mathcal{B}(x^-)$.

Proposition 5. *The Bellman function $\mathcal{B}$ is restrictively concave in the x-variables.*

Proof. Let $\varepsilon > 0$ be given and $x^\pm \in \Omega$. By the definition of $\mathcal{B}$, there exists $f^\pm, g^\pm$ on $I^\pm$ such that $\langle f \rangle_{J^\pm} = x_1^\pm, \langle g \rangle_{J^\pm} = x_2^\pm, \langle |f^\pm|^p \rangle_{I^\pm} = x_3^\pm$ and

$$\mathcal{B}(x^\pm) - \langle [(g^\pm)^2 + \tau^2 (f^\pm)^2]^{\frac{p}{2}} \rangle_{I^\pm} \leq \varepsilon.$$

On $I = I^+ \cup I^-$ we define f and g as $f := f^+ \chi_{I^+} + f^- \chi_{I^-}, g := g^+ \chi_{I^+} + g^- \chi_{I^-}$. So,

$$\begin{aligned} |x_1^+ - x_1^-| = |\langle f \rangle_{I^+} - \langle f \rangle_{I^-}| &= \Big| \frac{1}{|I^+|} \int_{|I^-|} f - \frac{1}{|I^-|} \int_{I^-} f \Big| \\ &= \Big| \frac{1}{\alpha^+ |I|} \int_{|I^-|} f - \frac{1}{\alpha^- |I|} \int_{I^-} f \Big| = \frac{1}{|I|} \Big| \int f \Big(\frac{1}{\alpha^+} \chi_{I^+} - \frac{1}{\alpha^-} \chi_{I^-} \Big) \Big| \end{aligned}$$

$$= \sqrt{\frac{|I|}{\alpha^+\alpha^-}} \left| \int f h_I \right| =: \sqrt{\frac{|I|}{\alpha^+\alpha^-}} \left|(f, h_I)\right|.$$

Similarly,$|x_2^+ - x_2^-| = \sqrt{\frac{|I|}{\alpha^+\alpha^-}} \left|(g, h_I)\right|$. So our assumption $|x_1^+ - x_1^-| = |x_2^+ - x_2^-|$ is equivalent to $|(f, h_I)| = |(g, h_I)|$. Since $x_1 = \langle f \rangle_I, x_2 = \langle g \rangle_I$ and $x_3 = \langle |f|^p \rangle_I$, f and g are test functions and so

$$\begin{aligned}
\mathcal{B}(x) &\geq \langle (g^2 + \tau^2 f^2)^{\frac{p}{2}} \rangle_I \\
&= \alpha^+ \langle [(g^+)^2 + \tau^2 (f^+)^2]^{\frac{p}{2}} \rangle_{I^+} + \alpha^- \langle [(g^-)^2 + \tau^2 (f^-)^2]^{\frac{p}{2}} \rangle_{I^-} \\
&\geq \alpha^+ \mathcal{B}(x^+) + \alpha^- \mathcal{B}(x^-) - \varepsilon.
\end{aligned}$$

□

At this point we do not quite have concavity of $\mathcal{B}$ on Ω since there is the restriction $|x_1^+ - x_1^-| = |x_2^+ - x_2^-|$ needed. To make this condition more manageable, we will make a change of coordinates. Let $y_1 := \frac{x_2+x_1}{2}, y_2 := \frac{x_2-x_1}{2}$ and $y_3 := x_3$. We will also change notation for the Bellman function and corresponding domain in the new variable y. Let $\mathcal{M}(y_1, y_2, y_3) := \mathcal{B}(x_1, x_2, x_3) = \mathcal{B}(y_1 - y_2, y_1 + y_2, y_3)$. Then the domain of definition for $\mathcal{M}$ will be $\Xi := \{y \in \mathbb{R}^3 : y_3 \geq 0, |y_1 - y_2|^p \leq y_3\}$.

If we consider $x^\pm \in \Omega$ such that $|x_1^+ - x_1^-| = |x_2^+ - x_2^-|$, then the corresponding points $y^\pm \in \Xi$ satisfy either $y_1^+ = y_1^-$ or $y_2^+ = y_2^-$. This implies that fixing y_1 as $y_1^+ = y_1^-$ or y_2 as $y_2^+ = y_2^-$ will make $\mathcal{M}$ concave with respect to y_2, y_3 under fixed y_1 and with respect to y_1, y_3 under y_2 fixed.

Rather than using Proposition 5 to check the concavity of the Bellman function we can just check it in the following way, assuming $\mathcal{M}$ is C^2. Let $j \neq i \in \{1, 2\}$ and fix y_i as

$y_i^+ = y_i^-$. Then $\mathcal{M}$ as a function of y_j, y_3 is concave if

$$\begin{pmatrix} \mathcal{M}_{y_j y_j} & \mathcal{M}_{y_j y_3} \\ \mathcal{M}_{y_3 y_j} & \mathcal{M}_{y_3 y_3} \end{pmatrix} \leq 0,$$

which is equivalent to

$$\mathcal{M}_{y_j y_j} \leq 0, \mathcal{M}_{y_3 y_3} \leq 0, D_j = \mathcal{M}_{y_j y_j}\mathcal{M}_{y_3 y_3} - \mathcal{M}_{y_3 y_j}\mathcal{M}_{y_j y_3} \geq 0.$$

Proposition 6. *(Restrictive Concavity in y-variables) Let $j \neq i \in \{1,2\}$ and fix y_i as $y_i^+ = y_i^-$. If $\mathcal{M}_{y_j y_j} \leq 0, \mathcal{M}_{y_3 y_3} \leq 0$ and $D_j = \mathcal{M}_{y_j y_j}\mathcal{M}_{y_3 y_3} - (\mathcal{M}_{y_j y_3})^2 \geq 0$ for $j = 1$ and $j = 2$, then $\mathcal{M}$ is Restrictively concave.*

The Bellman function, as it turns out, has many other nice properties.

Proposition 7. *Suppose that $\mathcal{M}$ is $C^1(\Xi)$, then $\mathcal{M}$ has the following properties.*

(i) Symmetry: $\mathcal{M}(y_1, y_2, y_3) = \mathcal{M}(y_2, y_1, y_3) = \mathcal{M}(-y_1, -y_2, y_3)$

(ii) Dirichlet boundary data: $\mathcal{M}(y_1, y_2, (y_1 - y_2)^p) = ((y_1 + y_2)^2 + \tau^2(y_1 - y_2)^2)^{\frac{p}{2}}$

(iii) Neumann conditions: $\mathcal{M}_{y_1} = \mathcal{M}_{y_2}$ *on* $y_1 = y_2$ *and* $\mathcal{M}_{y_1} = -\mathcal{M}_{y_2}$ *on* $y_1 = -y_2$

(iv) Homogeneity: $\mathcal{M}(ry_1, ry_2, r^p y_3) = r^p\mathcal{M}(y_1, y_2, y_3), \forall r > 0$

(v) Homogeneity relation: $y_1\mathcal{M}_{y_1} + y_2\mathcal{M}_{y_2} + py_3\mathcal{M}_{y_3} = p\mathcal{M}$

Proof. (i) Note that we get $\mathcal{B}(x_1, x_2, x_3) = \mathcal{B}(-x_1, x_2, x_3) = \mathcal{B}(x_1, -x_2, x_3)$ by considering test functions $\widetilde{f} = -f$ and $\widetilde{g} = -g$. Change coordinates from x to y and the result follows.

(ii) On the boundary $\{x_3 = |x_1|^p\}$ of Ω we see that

$$\frac{1}{|I|}\int_I |f|^p = \langle |f|^p \rangle_I = x_3 = |x_1|^p = |\langle f \rangle_I|^p = \left|\frac{1}{|I|}\int_I f\right|^p$$

is only possible if $f \equiv$ const. (i.e. $f = x_1$). But, $|(f, h_J)| = |(g, h_J)|$ for all $J \in \mathcal{I}$, which implies that $g \equiv$ const. (i.e. $g = x_2$). Then $\mathcal{B}(x_1, x_2, |x_1|^p) = \langle (g^2+\tau^2 f^2)^{\frac{p}{2}} \rangle_I = (x_2^2+\tau^2 x_1^2)^{\frac{p}{2}}$. Changing coordinates gives the result. If we are not on the boundary, but close to it, then any test function has the p-th power of its average close to the average of its p-th power. This means that the function itself is close to its average in L^p-norm. We know a priori that the martingale transform is bounded in L^p, $1 < p < \infty$. Hence the value of $\mathcal{B}$ for the data close to the boundary will be close to the martingale transform of a constant function considered above. We are done.

(iii) This follows from from (i).

(iv) Consider the test functions $\widetilde{f} = rf, \widetilde{g} = rg$

(v) Differentiate (iv) with respect to r and evaluate it at $r = 1$. □

Now that we have all of the properties of the Bellman function we will turn our attention to actually finding it. Proposition 6 gives us two partial differential inequalities to solve, $D_1 \geq 0, D_2 \geq 0$, that the Bellman function must satisfy. Since the Bellman function is the supremum of the left-hand side of our estimate under the condition that g is the martingale transform of f and must also satisfy the estimates in Proposition 6, it seems reasonable that the Bellman function (being the optimal such function) may satisfy the following, for either $j = 1$ or $j = 2$:

$$D_j = \mathcal{M}_{y_j y_j}\mathcal{M}_{y_3 y_3} - (\mathcal{M}_{y_3 y_j})^2 = 0.$$

The PDE that we now have is the well known Monge-Ampère equation which has a solution. Let us make it clear that we have added an assumption.

Assumption 8. $D_j = \mathcal{M}_{y_j y_j}\mathcal{M}_{y_3 y_3} - (\mathcal{M}_{y_3 y_j})^2 = 0$, *for either* $j = 1$ *or* $j = 2$.

Adding this assumption comes with a price. Any function that we construct, satisfying all properties of the Bellman function, must somehow be shown to be the Bellman function. We will refer to any function satisfying some, or all Bellman function properties as a Bellman function candidate.

In Subsection 2.1.4 we use [35] to explain what consequences the assumption 8 will have on our search of Bellman function candidate.

2.1.4 Monge-Ampère equation and method of "characteristics"

Let $v = v(x_1, ..., x_n)$ is a smooth function satisfying the following Monge-Ampère equation in some domain Ω

$$\det d^2 v = \det \begin{pmatrix} v_{x_1 x_1} & \cdots & v_{x_1 x_n} \\ \multicolumn{3}{c}{\dotfill} \\ v_{x_n x_1} & \cdots & v_{x_n x_n} \end{pmatrix} = 0\,, \qquad \forall x = (x_1, \ldots, x_n) \in \Omega\,, \tag{2.2}$$

and suppose that this matrix has rank $n-1$; i.e., all smaller minors of $d^2 v$ are non-zero. Then there are functions $t_i(x_1, \ldots, x_n)$, $i = 0, 1, \ldots, n$, such that

$$v(x) = t_0 + t_1 x_1 + t_2 x_2 + \cdots + t_n x_n \tag{2.3}$$

and the following $n-1$ linear equations hold:

$$dt_0 + x_1 dt_1 + x_2 dt_2 + \cdots + x_{n-1} dt_{n-1} + x_n dt_n = 0\,. \tag{2.4}$$

Let us explain why this is $n-1$ equations and why they are linear. One needs to read (2.4) as follows: we think that, say, $t_1, \ldots, t_{n-1}$ are $n-1$ independent variables and t_n, t_0 are functions of them. Then (2.4) can be rewritten as

$$\Big(\frac{\partial t_0}{\partial t_1} + x_1 + x_n \frac{\partial t_n}{\partial t_1}\Big) dt_1 + \cdots + \Big(\frac{\partial t_0}{\partial t_{n-1}} + x_{n-1} + x_n \frac{\partial t_n}{\partial t_{n-1}}\Big) dt_{n-1} = 0\,,$$

whence

$$x_i + x_n \frac{\partial t_n}{\partial t_i} + \frac{\partial t_0}{\partial t_i} = 0\,, \quad i = 1, \ldots, n-1\,.$$

So we get $n-1$ equations.

Remark. In general we can choose **any** $n-1$ variables as independent, of course. Since the order of variables is arbitrary, sometimes the first $n-1$ is not the most convenient choice.

Now why are these linear equations? We think that $t_1, \ldots, t_{n-1}$ is fixed. Then the $n-1$ equations give us linear relationships in $x_1, \ldots, x_n$, so $n-1$ hyperplanes.

Therefore, (2.4) gives the intersection of $n-1$ hyperplanes, so gives us a line. We can call it $L_{t_1,\ldots,t_{n-1}}$. These lines foliate domain Ω and (2.3) shows that v is a linear function on each such line.

Let us prove all these propositions. Matrix $d^2 v$ annihilates one vector $\Theta(x)$ at every $x = (x_1, ..., x_n) \in \Omega$. So we get a vector field Θ. Consider its integral curve $x_1 =$

$x_1(x_n), \ldots, x_{n-1} = x_{n-1}(x_n)$. Vector $\Theta(x)$ is a tangent vector to that curve; i.e.,

$$\Theta = \Theta_n \begin{pmatrix} x_1' \\ x_2' \\ \ldots \\ x_{n-1}' \\ 1 \end{pmatrix}. \tag{2.5}$$

Consider a new function $g(x_n) = v(x_1(x_n), \ldots, x_{n-1}(x_n), x_n)$. Due to (2.5) its second derivative is

$$g'' = \left\langle d^2 v \begin{pmatrix} x_1' \\ x_2' \\ \ldots \\ x_{n-1}' \\ 1 \end{pmatrix}, \begin{pmatrix} x_1' \\ x_2' \\ \ldots \\ x_{n-1}' \\ 1 \end{pmatrix} \right\rangle + v_{x_1} x_1'' + \cdots + v_{x_{n-1}} x_{n-1}'' = v_{x_1} x_1'' + \cdots + v_{x_{n-1}} x_{n-1}''. \tag{2.6}$$

Now, let us also show that $v_{x_1}, \ldots, v_{x_{n-1}}$ are constants on this integral curve. Suppose we are standing on the integral curve. The surface $v_{x_1} = t_1 =$ const. has normal $(v_{x_1 x_1}, \ldots, v_{x_1 x_n})$, that is the first row of matrix $d^2 v$, which is orthogonal to the directional vector Θ of the integral curve. Hence Θ is in the tangent hyperplane to the surface $v_{x_1} = t_1$. The same is true for the surfaces $v_{x_i} = t_i =$ const., $i = 2, \ldots, n-1$. Intersection of these surfaces gives us our integral curves, because Θ is in the intersection of all tangent planes to these surfaces. Therefore the curves $C_{t_1, \ldots, t_{n-1}}$ enumerated by constants $t_1, \ldots, t_n$ are just the integral curves of the tangent bundle Θ. Thus, (2.6) can be rewritten as

$$\frac{d^2}{dx_n^2}\big(g - (t_1\, x_1 + \cdots + t_n\, x_{n-1})\big) = 0. \tag{2.7}$$

We obtain that the second derivative of a function $g(x_n) = v(x_1(x_n), \ldots, x_{n-1}(x_n), x_n)$ in (2.7) is zero. So function this function is linear in x_n, that is $g(x_n) = t_n x_n + t_0$, where the constants t_n, t_0 depend only on the curve $C_{t_1,\ldots,t_{n-1}}$, that is

$$t_n = t_n(t_1, \ldots, t_{n-1})\,, \qquad t_0 = t_0(t_1, \ldots, t_{n-1})\,.$$

Looking at the definition of $g(x_n)$ we see that we obtained on $C_{t_1,\ldots,t_{n-1}}$ the following

$$v(x_1(x_n), \ldots, x_{n-1}(x_n), x_n) = t_0 + x_1 t_1 + \cdots + x_{n-1} t_{n-1} + t_n x_n\,.$$

Since we assumed our vector field to be smooth and its integral curves foliate the whole domain, varying parameters $t_1, \ldots, t_n$ we get (2.3). To check (2.4) take a full differential in (2.3). Then

$$[v_{x_1} dx_1 + \cdots + v_{x_n} dx_n] = dv = [dt_0 + t_1 dx_1 + \cdots + t_n dx_n] + x_1 dt_1 + \cdots + x_n dt_n\,. \tag{2.8}$$

We are on $C_{t_1,\ldots,t_{n-1}}$ and so $v_{x_i} = t_i$, $i = 1, \ldots, n-1$ as we established already. But it is also easy to see that

$$v_{x_n} = t_n(t_1, \ldots, t_{n-1})$$

on $C_{t_1,\ldots,t_{n-1}}$. In fact, all v_{x_i} and all t_i are symmetric. We could have chosen to represent the integral curve of Θ not as $x_i = x_i(x_n), i = 1, \ldots, n-1$ but as $x_j = x_j(x_1)$, $j = 2, \ldots, n$. Now we see that two expression in brackets in (2.8) are equal. Then we obtain

$x_1dt_1 + \cdots + x_ndt_n + dt_0 = 0$, which is the desired $n-1$ linear relationships (2.4).

From the consideration above we can see that the following proposition holds.

Proposition 9. *For $j = 1$ or 2, if $\mathcal{M}_{y_jy_j}\mathcal{M}_{y_3y_3} - (\mathcal{M}_{y_3y_j})^2 = 0$ has a smooth solution, then it should have a form $M(y) = y_jt_j + y_3t_3 + t_0$. Here $t_i = t_i(y_j, y_3)$, $i = 0, 1, 2$ are functions of y_j, y_3 (and parameter y_{3-j}). Furthermore, t_0, t_j, t_3 have common level sets called characteristics, and the differentials of these functions satisfy $y_jdt_j + y_3dt_3 + dt_0 = 0$. This shows that these common level sets (characteristics) are straight lines in the $y_j \times y_3$ plane. Moreover, one can choose functions t_j, t_3 as follows: $t_j = M_{y_j}, t_3 = M_{y_3}$.*

This is a result of Pogorelov, see [33]. We explained it above in Subsection 8. Now that we have a solution M to the Monge-Ampère, we need get rid of t_0, t_j, t_3 so that we have an explicit form of M, without the characteristics. We note that a solution to the Monge-Ampère is not necessarily the Bellman function. It must satisfy the restrictive concavity of Proposition 6, be C^1-smooth, and satisfy the properties of Proposition 7. The restrictive concavity property is one of the key deciding factors of whether or not we have a Bellman function in many cases. Even if the Monge-Ampère solution satisfies all of those conditions, it must still be shown to be equal to the Bellman function, because we added an additional assumption (Assumption 8) to get the Monge-Ampère solution as a starting point. This will be considered rigorously in Section 2.3, after we obtain a solution to the Monge-Ampère equation, with the appropriate Bellman function properties. So from this point on we will use M and B to denote solutions to the Monge-Ampère equation; i.e. Bellman function candidates, and $\mathcal{M}$ and $\mathcal{B}$ to denote the true Bellman function.

2.2 Computing the Bellman function candidate from the Monge-Ampère solution

Due to the symmetry property of $\mathcal{M}$, from Proposition 7, we only need to consider a portion of the domain Ξ, which we will denote as, $\Xi_+ := \{y : -y_1 \leq y_2 \leq y_1, y_3 \geq 0, (y_1 - y_2)^p \leq y_3\}$. Since the characteristics are straight lines, one end of each line must be on the boundary. Let U denote the point at which the characteristic touches the part of the boundary $\{y : (y_1 - y_2)^p = y_3\}$ (if it does). Furthermore, the characteristics can only behave in one of the following four ways, since they are straight lines in the plane:

(1) The characteristic goes from U to $\{y : y_1 = y_2\}$

(2) The characteristic goes from U to to infinity, running parallel to the y_3-axis

(3) The characteristic goes from U to $\{y : y_1 = -y_2\}$

(4) The characteristic goes from U to $\{y : (y_1 - y_2)^p = y_3\}$

(5) The characteristic goes from the "wall" $y_2 = -y_1$ to the "wall" $y_2 = y_1$

To find a Bellman function candidate we must first fix a variable (y_1 or y_2) and a case for the characteristics. Then we use the Bellman function properties to get rid of the characteristics. If the Monge-Ampère solution satisfies restrictive concavity, then it is a Bellman function candidate. However, checking the restrictive concavity is quite difficult in many of the cases, since it amounts to doing second derivative estimates for an implicitly defined function. Let us now find our Bellman function candidate.

Remark 10. Since we will have either y_1 or y_2 fixed in each case, there will be eight cases in all. Let $(1_j), (2_j), (3_j), (4_j), (5_j)$ denote the case when $M_{y_j y_j} M_{y_3 y_3} - (M_{y_3 y_j})^2 = 0$ and

y_i is fixed, where $i \neq j$. Also, we will denote $G(z_1, z_2) := (z_1 + z_2)^{p-1}[z_1 - (p-1)z_2]$ and $\omega := \left(\frac{\mathcal{M}(y)}{y_3}\right)^{\frac{1}{p}}$ from this point on.

Remark 11. Characteristics foliate the domain. Therefore the case of the existence of characteristic of the type (5_j) automatically implies that there exists a characteristic of type (4_j). Therefore we are not looking at the case (5_j) at all in what follows.

2.2.1 Bellman candidate for $2 < p < \infty$

The solution to the Monge-Ampère equation when $2 < p < \infty$, is only partially valid on the domain in two cases, due to restrictive concavity. Case (1_2), will give us an implicit solution that is valid on part of Ξ_+ and Case (2_2) will give us an explicit solution for the remaining part of Ξ_+. First, we deal with Case (1_2).

2.2.1.1 Case (1_2)

Since we are considering Case (1_2), $y_1 \geq 0$ is fixed until the point that we have the implicit solution independent of the characteristics satisfying all of the Bellman function properties.

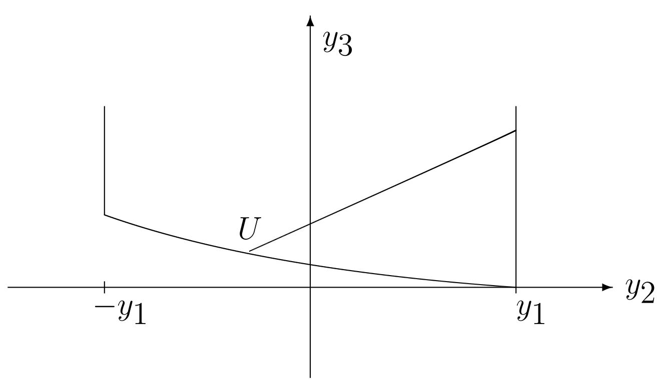

Figure 2.1: Sample characteristic of solution from Case (1_2)

Proposition 12. *For* $1 < p < \infty$ *and* $\frac{p-2}{p} y_1 < y_2 < y_1$, M *is given implicitly by the relation* $G(y_1 + y_2, y_1 - y_2) = y_3 G(\sqrt{\omega^2 - \tau^2}, 1)$, *where* $G(z_1, z_2) := (z_1 + z_2)^{p-1}[z_1 - (p-1)z_2]$ *on* $z_1 + z_2 \geq 0$ *and* $\omega := \left(\frac{M(y)}{y_3}\right)^{\frac{1}{p}}$.

This is proven through a series of Lemmas.

Lemma 13. $M(y) = t_2 y_2 + t_3 y_3 + t_0$ *on the characteristic* $y_2 dt_2 + y_3 dt_3 + dt_0 = 0$ *can be simplified to* $M(y) = \left(\frac{\sqrt{(y_1+u)^2+\tau^2(y_1-u)^2}}{y_1-u}\right)^p y_3$, *where* u *is the unique solution to the equation* $\frac{y_2+(\frac{2}{p}-1)y_1}{y_3} = \frac{u+(\frac{2}{p}-1)y_1}{(y_1-u)^p}$ *and* $\frac{p-2}{p} y_1 < y_2 < y_1$

Proof. A characteristic in Case (1_2) is from $U = (y_1, u, (y_1 - u)^p)$ to $W = (y_1, y_1, w)$. Throughout the proof we will use the properties of the Bellman function from Proposition 7. Using the Neumann property and the property from Proposition 9 we get $M_{y_1} = M_{y_2} = t_2$ at W. By homogeneity at W we get

$$py_2 t_2 + pwt_3 + pt_0 = pM(W) = y_1 M_{y_1} + y_2 M_{y_2} + py_3 M_{y_3} = 2y_1 t_2 + pwt_3.$$

Then $t_0 = (\frac{2}{p} - 1)y_1 t_2$ and $dt_0 = (\frac{2}{p} - 1)y_1 dt_2$, since y_1 is fixed. So $M(y) = [y_2 + (\frac{2}{p} - 1)y_1]t_2 + y_3 t_3$ on $[y_2 + (\frac{2}{p} - 1)y_1]dt_2 + y_3 dt_3 = 0$. By substitution we get, $M(y) = y_3[t_3 - t_2 \frac{dt_3}{dt_2}]$ on characteristics. But, $t_2, t_3, \frac{dt_3}{dt_2}$ are constant on characteristics, which gives that $\frac{M(y)}{y_3} \equiv$ const. as well. We can calculate the value of the constant by using the Dirichlet boundary data for M at U. Therefore, $M(y) = \left(\frac{\sqrt{(y_1+u)^2+\tau^2(y_1-u)^2}}{y_1-u}\right)^p y_3$, where u is the solution to the equation

$$\frac{y_2 + (\frac{2}{p} - 1)y_1}{y_3} = \frac{u + (\frac{2}{p} - 1)y_1}{(y_1 - u)^p}. \tag{2.9}$$

Now fix $u = -(\frac{2}{p} - 1)y_1$. Then we see that $y_2 = -(\frac{2}{p} - 1)y_1 = u$ is also fixed by (2.9). This means that the characteristics are limited to part of the domain, as shown in Figure 2.2, since they start at U and end at $W \in \{y_1 = y_2\}$. All that remains is verifying the equation

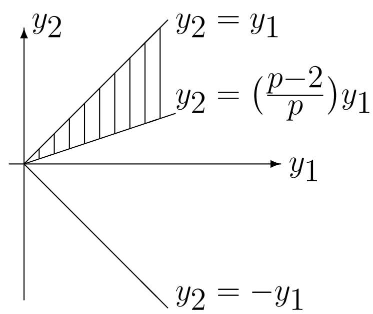

Figure 2.2: Sector for characteristics in Case (1_2), when $p > 2$.

(2.9) has exactly one solution $u = u(y_1, y_2, y_3)$ in the sector $\frac{p-2}{p}y_1 < y_2 < y_1$. Indeed, the function

$$f(u) := y_3\left[u + \left(\frac{2}{p} - 1\right)y_1\right] - (y_1 - u)^p\left[y_2 + \left(\frac{2}{p} - 1\right)y_1\right]$$

is monotone increasing for $u < y_1$, $f(-(\frac{2}{p} - 1)y_1) = -(\frac{2}{p}y_1)^p\Big[y_2 + (\frac{2}{p} - 1)y_1\Big] < 0$ and $f(y_1) = \frac{2}{p}y_1y_3 > 0$. Therefore, we do get a unique solution, u, in the sector. $\square$

Lemma 14. *$M(y) = \left(\frac{\sqrt{(y_1+u)^2+\tau^2(y_1-u)^2}}{y_1-u}\right)^p y_3$ can be rewritten as $G(y_1 + y_2, y_1 - y_2) = y_3G(\sqrt{\omega^2 - \tau^2}, 1)$ for $\frac{p-2}{p}y_1 < y_2 < y_1$.*

Proof. $\omega = \left(\frac{M(y)}{y_3}\right)^{\frac{1}{p}} = \frac{\sqrt{(y_1+u)^2+\tau^2(y_1-u)^2}}{y_1-u} \geq |\tau|\frac{|y_1-u|}{y_1-u} = |\tau|$.

Since $y_1 \pm u \geq 0$ and $\omega^2 - \tau^2 \geq 0$, $u = \frac{\sqrt{\omega^2-\tau^2}-1}{\sqrt{\omega^2-\tau^2}+1}y_1$ by inversion. Substituting this into $\frac{y_2+(\frac{2}{p}-1)y_1}{y_3} = \frac{u+(\frac{2}{p}-1)y_1}{(y_1-u)^p}$ gives

$$2^{p-1}y_1^{p-1}[py_2 - (p-2)y_1] = y_3\left(\sqrt{\omega^2 - \tau^2} + 1\right)^{p-1}\left[\sqrt{\omega^2 - \tau^2} - (p-1)\right]$$

or $(x_1 + x_2)^{p-1}[x_2 - (p-1)x_1]$

$$= \left[\sqrt{B^{\frac{2}{p}} - \left(\tau x_3^{\frac{1}{p}}\right)^2} + x_3^{\frac{1}{p}}\right]^{p-1} \left[\sqrt{B^{\frac{2}{p}} - \left(\tau x_3^{\frac{1}{p}}\right)^2} - (p-1)x_3^{\frac{1}{p}}\right].$$

Thus,

$$G(x_2, x_1) = G\left(\sqrt{B^{\frac{2}{p}} - \left(\tau x_3^{\frac{1}{p}}\right)^2}, x_3^{\frac{1}{p}}\right) \text{ or } G(y_1 + y_2, y_1 - y_2) = y_3 G(\sqrt{\omega^2 - \tau^2}, 1). \ \Box$$

This proves Proposition 12. We have constructed a partial Bellman function candidate from the Monge-Ampère solution in Case(1_2), so y_1 no longer needs to be fixed. All of the properties of the Bellman function were used to derive this partial Bellman candidate, but the restrictive concavity from Proposition 6 still needs to be verified. To verify restrictive concavity, we need that $M_{y_2 y_2} \leq 0, M_{y_3 y_3} \leq 0, D_2 \geq 0$ and $M_{y_1 y_1} \leq 0, D_1 \geq 0$. By assumption $D_2 = 0$, so we need not worry about that estimate. The remaining estimates will be verified in a series of Lemmas. The first Lemma is an idea taken from Burkholder [13] to make the calculations for computing mixed partials shorter. In the Lemma, we compute the partials of arbitrary functions which we will choose specifically later, although it is not hard to see what the appropriate choices should be.

Lemma 15. *Let* $H = H(y_1, y_2), \Phi(\omega) = \frac{H(y_1,y_2)}{y_3}, R_1 = R_1(\omega) := \frac{1}{\Phi'}$ *and* $R_2 = R_2(\omega) := R_1' = -\frac{\Phi''}{\Phi'^2}$. *Then*

$$M_{y_3 y_3} = \frac{p\omega^{p-2} R_1 H^2}{y_3^3}[\omega R_2 + (p-1)R_1]$$

$$M_{y_3 y_i} = -\frac{p\omega^{p-2}R_1 H H'}{y_3^2}[\omega R_2 + (p-1)R_1]$$
$$M_{y_i y_i} = \frac{p\omega^{p-2}R_1}{y_3}\left([\omega R_2 + (p-1)R_1](H')^2 + \omega y_3 H''\right)$$
$$D_i = M_{y_3 y_3}M_{y_i y_i} - M^2_{y_3 y_i} = \frac{p^2\omega^{2p-3}R_1^2 H^2 H''}{y_3^3}[\omega R_2 + (p-1)R_1],$$

where $\omega = \dfrac{M(y)}{y_3}$.

Proof. First of all we calculate the partial derivatives of ω:

$$\Phi'\omega_{y_3} = -\frac{H}{y_3^2} \quad\Longrightarrow\quad \omega_{y_3} = -\frac{R_1 H}{y_3^2}\,,$$
$$\Phi'\omega_{y_i} = \frac{H_{y_i}}{y_3} \quad\Longrightarrow\quad \omega_{y_i} = \frac{R_1 H_{y_i}}{y_3} = \frac{R_1 H'}{y_3}, \qquad i = 1, 2\,.$$

Here and further we shall use notation H' for any partial derivative H_{y_i}, $i = 1, 2$. This cannot cause any confusion since only one $i \in \{1, 2\}$ participates in the calculation of D_i.

$$\omega_{y_3 y_3} = -\frac{R_2\omega_{y_3}H}{y_3^2} + 2\frac{R_1 H}{y_3^3} = \frac{R_1 H}{y_3^4}(R_2 H + 2y_3)\,,$$
$$\omega_{y_3 y_i} = -\frac{R_2\omega_{y_i}H}{y_3^2} - \frac{R_1 H'}{y_3^2} = -\frac{R_1 H'}{y_3^3}(R_2 H + y_3)\,,$$
$$\omega_{y_i y_i} = \frac{R_2\omega_{y_i}H}{y_3} + \frac{R_1 H'}{y_3} = \frac{R_1}{y_3^2}(R_2(H')^2 + y_3 H'')\,.$$

Now we pass to the calculation of derivatives of $M = y_3\omega^p$:

$$M_{y_3} = p y_3 \omega^{p-1}\omega_{y_3} + \omega^p\,,$$
$$M_{y_i} = p y_3 \omega^{p-1}\omega_{y_i}\,;$$

$$
\begin{aligned}
M_{y_3y_3} &= py_3\omega^{p-1}\omega_{y_3y_3} + 2p\omega^{p-1}\omega_{y_3} + p(p-1)y_3\omega^{p-2}\omega_{y_3}^2 \\
&= \frac{p\omega^{p-2}R_1H^2}{y_3^3}[\omega R_2 + (p-1)R_1]\,, \qquad (2.10) \\
M_{y_3y_i} &= py_3\omega^{p-1}\omega_{y_3y_i} + p\omega^{p-1}\omega_{y_i} + p(p-1)y_3\omega^{p-2}\omega_{y_3}\omega_{y_i} \\
&= -\frac{p\omega^{p-2}R_1HH'}{y_3^2}[\omega R_2 + (p-1)R_1]\,, \\
M_{y_iy_i} &= py_3\omega^{p-1}\omega_{y_iy_i} + p(p-1)y_3\omega^{p-2}\omega_{y_i}^2 \\
&= \frac{p\omega^{p-2}R_1}{y_3}\Big([\omega R_2 + (p-1)R_1](H')^2 + \omega y_3H''\Big)\,. \qquad (2.11)
\end{aligned}
$$

This yields

$$
D_i = M_{y_3y_3}M_{y_iy_i} - M_{y_3y_i}^2 = \frac{p^2\omega^{2p-3}R_1^2H^2H''}{y_3^3}[\omega R_2 + (p-1)R_1]\,. \qquad (2.12)
$$

□

Lemma 16. *If $\alpha_i, \beta_i \in \{\pm 1\}$ and $H(y_1, y_2) = G(\alpha_1y_1 + \alpha_2y_2, \beta_1y_1 + \beta_2y_2)$, then*

$$
H'' = \begin{cases} 4G_{z_1z_2}, & \alpha_j = \beta_j \\ 0, & \alpha_j = -\beta_j. \end{cases}
$$

Consequently, in Case (1_2), $\operatorname{sign} H'' = -\operatorname{sign}(p-2)$.

Proof.

$$
\begin{aligned}
H'' &= \frac{\partial^2}{\partial y_i^2}G(\alpha_1y_1 + \alpha_2y_2, \beta_1y_1 + \beta_2y_2) \\
&= \alpha_i^2G_{z_1z_1} + 2\alpha_i\beta_iG_{z_1z_2} + \beta_i^2G_{z_2z_2} \\
&= G_{z_1z_1} + G_{z_2z_2} \pm 2G_{z_1z_2}\,,
\end{aligned}
$$

where the "+" sign has to be taken if the coefficients in front of y_i are equal and the "−" sign in the opposite case.

The derivatives of G are simple:

$$G_{z_1} = p(z_1 + z_2)^{p-2}\left[z_1 - (p-2)z_2\right],$$
$$G_{z_2} = -p(p-1)z_2(z_1 + z_2)^{p-2};$$

$$G_{z_1 z_2} = p(p-1)(z_1 + z_2)^{p-3}\left[z_1 - (p-3)z_2\right],$$
$$G_{z_1 z_2} = -p(p-1)(p-2)z_2(z_1 + z_2)^{p-3},$$
$$G_{z_2 z_2} = -p(p-1)(z_1 + z_2)^{p-3}\left[z_1 + (p-1)z_2\right].$$

Note that $G_{z_1 z_1} + G_{z_2 z_2} = 2G_{z_1 z_2}$, and therefore,

$$H'' = \begin{cases} 4G_{z_1 z_2}, & \alpha_j = \beta_j \\ 0, \alpha_j = -\beta_j \end{cases}$$

Now in Case (1_2), we must choose $\alpha_1 = 1, \alpha_2 = 1, \beta_1 = 1$ and $\beta_2 = -1$ for H to match how the implicit solution was defined in terms of G in Proposition 12. Then $G_{z_1 z_2} = -p(p-1)(p-2)(y_1 - y_2)(2y_1)^{p-3}$. □

Remark 17. Let $\beta := \sqrt{\omega^2 - \tau^2}$ from this point on. In Case $(1_2), \beta > p-1$ in the sector $\frac{p-2}{p} y_1 < y_2 < y_1$. Equivalently, $B > (\tau^2 + (p-1)^2)^{\frac{p}{2}}$ in $\frac{p-2}{p} y_1 < y_2 < y_1$.

This is an easy application of Proposition 12:

$$(\beta+1)^{p-1}[\beta-p+1] = G(\beta,1) = \frac{1}{y_3}G(y_1+y_2, y_1-y_2)$$
$$= (2y_1)^{p-1}[-(p-2)y_1+py_2] > 0.$$

Before we can compute the signs of $M_{y_1y_1}, M_{y_2y_2}, M_{y_3y_3}$ and D_1 we need a technical lemma.

Lemma 18. *If* $1<p<\infty$ *and* $\tau\in\mathbb{R}$, *then*

$$g(\beta) := -p(p-2)\omega\beta^{-3}(\beta+1)^{p-3}[(\tau^2+p-1)\beta^2-\tau^2(p-3)\beta+\tau^2]$$

satisfies $\operatorname{sign} g(\beta) = -\operatorname{sign}(p-2)$ *in Case* (1_2).

Proof. The only terms controlling the sign in g are $(p-2)$ and the quadratic part, which we will denote $q(\beta)$. So all that is needed is to simply determine the sign of q. The discriminant of q is $\tau^2(p-1)[\tau^2(p-5)-4]$.

If $p\le 5$, then the discriminant of q is negative and so $q(\beta)>0$. If $p>5$ and $\tau^2(p-5)-4<0$, then $q(\beta)>0$ once again.

The only case left to consider is if $p>5$ and $\tau^2(p-5)-4\ge 0$. The zeros of q are given by $\beta = \frac{\tau^2(p-3)\pm|\tau|\sqrt{p-1}\sqrt{\tau^2(p-5)-4}}{2(\tau^2+p-1)}$. Let β_1,β_2 be the zeros such that $\beta_2\ge\beta_1$. We claim that $\max\{p-1,\beta_2\}=p-1$. Indeed, $p-1-\beta_2>0$

$$\iff (p+1)\tau^2+2(p-1)^2 > |\tau|\sqrt{p-1}\sqrt{\tau^2(p-5)-4}$$
$$\iff 4(p-1)^4+4\tau^2(p+1)(p-1)^2+\tau^4(p+1)^2 > \tau^2(p-1)(\tau^2(p-5)-4)$$

$$\iff (p-1)^4 + \tau^2 p^2 (p-1) + \tau^4 (2p-1) > 0,$$

which is obviously true for all $\tau \in \mathbb{R}$. Now that we have proven the claim, recall that $\beta > p-1$, as shown in Remark 17. Therefore, $\beta > \beta_2$, so $q(\beta) > 0$ in this case. □

Lemma 19. *$D_1 > 0$ in Case (1_2) for all $\tau \in \mathbb{R}$.*

Proof. We use the partial derivatives of G computed in the proof of Lemma 16 to make the computations of Φ' and Φ'' easier.

$$\begin{aligned}
\Phi(\omega) &= G(\beta, 1) \\
\Phi'(\omega) &= p\omega[\beta+1]^{p-2}[1-(p-2)\beta^{-1}] \qquad (2.13) \\
\Phi''(\omega) &= p(\beta+1)^{p-2}[1-(p-2)\beta^{-1}] + p(p-2)\omega^2\beta^{-1}[\beta+1]^{p-3}[1-(p-2)\beta^{-1}] \\
&\quad + p(p-2)\omega^2\beta^{-3}[\beta-1]^{p-2} \\
\Lambda &= (p-1)\Phi' - \omega\Phi'' \\
&= p(p-2)\omega\beta^{-1}(\beta+1)^{p-2}[1-(p-2)\beta^{-1}] \\
&\quad - p(p-2)\omega^3\beta^{-3}(\beta+1)^{p-3}[\beta(\beta-p+2)+\beta+1] \\
&= p(p-2)\omega\beta^{-2}(\beta+1)^{p-3}[\beta-p+2]\{\beta(\beta+1)-\omega^2\} - p(p-2)\omega^3\beta^{-3}(\beta+1)^{p-2} \\
&= p(p-2)\omega\beta^{-3}[\beta(\beta-p+2)(\beta-\tau^2)-\omega^2(\beta+1)] \\
&= -p(p-2)\omega\beta^{-3}(\beta+1)^{p-3}[(\tau^2+p-1)\beta^2 - \tau^2(p-3)\beta + \tau^2]
\end{aligned}$$

So we can see that $\operatorname{sign}\Lambda = \operatorname{sign} g(\beta) = -\operatorname{sign}(p-2)$, by Lemma 18. Therefore, $\operatorname{sign} D_1 = \operatorname{sign} H'' \operatorname{sign}\Lambda = [-\operatorname{sign}(p-2)]^2$ by (2.12) and Lemma 16. □

Since $D_1 > 0$, then all that remains to be checked, for the restrictive concavity of M, is that $M_{y_iy_i}$ (for $i = 1, 2$) and $M_{y_3y_3}$ have the appropriate signs. But, it turns out that only for $2 < p < \infty$, will these have the appropriate signs.

Lemma 20. $\operatorname{sign} M_{y_1y_1} = \operatorname{sign} M_{y_2y_2} = \operatorname{sign} M_{y_3y_3} = -\operatorname{sign}(p-2)$ *in Case* (1_2) *for all* $\tau \in \mathbb{R}$. *Therefore, M is a partial Bellman function candidate for $2 < p < \infty$ but not for $1 < p < 2$, since it does not satisfy the required restrictive concavity.*

Proof. By (2.10),

$$M_{y_3y_3} = \frac{p\omega^{p-2}R_1^2H^2}{y_3^3}\left[\frac{\Lambda}{\Phi'}\right]$$

Remark 17 gives $\Phi' > 0$. From Lemma 18, $\operatorname{sign} M_{y_3y_3} = \operatorname{sign}\Lambda = \operatorname{sign} g(\beta) = -\operatorname{sign}(p-2)$. By (2.11), for $i = 1$or 2,

$$\begin{aligned} M_{y_iy_i} &= \frac{p\omega^{p-2}R_1}{y_3}\left[(\omega R_2 + (p-1)R_1)(H')^2 + \omega y_3 H''\right] \\ &= \frac{p\omega^{p-2}}{y_3(\Phi')^3}\left[\Lambda(H')^2 + \omega y_3 H''(\Phi')^2\right], \end{aligned}$$

giving $\operatorname{sign} M_{y_2y_2} = -\operatorname{sign}(p-2)$. □

The previous two lemmas established that the partial Bellman function candidate, from Case (1_2) satisfies the restrictive concavity property, for $2 < p < \infty$. The candidate was constructed using the remaining Bellman function properties, so it is in fact a partial candidate. Now we will turn our attention to Case (2). As it turns out, Case (2_2) also gives a partial Bellman function candidate, which, as luck would have it, is the missing half of the partial Bellman candidate just constructed.

2.2.1.2 Case (2_2) for $2 < p < \infty$

We can obtain a Bellman candidate from Case (2) without having to separately fix y_1 or y_2. Let us compute the solution in this case.

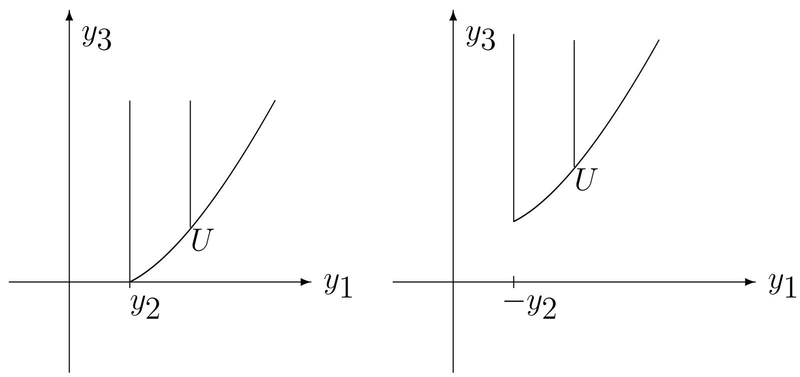

Figure 2.3: Sample characteristic of Monge-Ampère solution in Case (2_1)

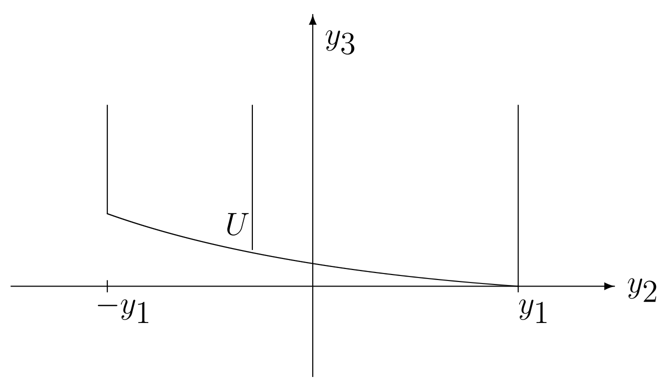

Figure 2.4: Sample characteristic of Monge-Ampère solution for Case (2_2)

Lemma 21. *In Case* (2) *we obtain*

$$M(y) = (1+\tau^2)^{\frac{p}{2}}[y_1^2 + 2\gamma y_1 y_2 + y_2^2]^{\frac{p}{2}} + c[y_3 - (y_1 - y_2)^p] \tag{2.14}$$

as a Bellman function candidate, where $c > 0$ is some constant and $\gamma = \frac{1-\tau^2}{1+\tau^2}$.

Proof. In Case (2), on the characteristic $y_i dt_i + y_3 dt_3 + dt_0 = 0$, y_1 and y_2 are fixed. Furthermore, on the characteristic, t_0, t_i, t_3 are fixed, so we have

$$\begin{aligned}
M(y) &= y_i t_i + y_3 t_3 + t_0 \\
&= (y_i t_i + t_0) + y_3 t_3 \\
&= c_1(y_1, y_2) + c_2(y_1, y_2) y_3
\end{aligned}$$

Then $M_{y_3 y_3} = 0$ and $M_{y_3 y_i} = \partial_{y_i} c_2$. Recall that $D_i \geq 0$ by Remark 6, so $\partial_{y_i} c_2(y_1, y_2) = 0$. This implies that c_2 is a constant. Using the boundary data from Proposition 7 gives $((y_1+y_2)^2+\tau^2(y_1-y_2)^2)^{\frac{p}{2}} = M(y_1, y_2, (y_1-y_2)^p) = c_1(y_1, y_2)+c_2(y_1-y_2)^p$. Solving for $c_1(y_1, y_2)$ gives the result. To see that $c_2 > 0$, just notice that as $y_3 \to \infty, M(y) \to \infty$. $\square$

It is not possible to determine if this Bellman function candidate satisfies restrictive concavity, unless we know the value of the constant c in Lemma 21. This constant can be computed by using the fact that (2.14) must agree with the partial candidate in Case (1_2) at $y_2 = \frac{p-2}{p} y_1$, if (2.14) is in fact a candidate itself.

Lemma 22. *In Case* (2_2), *the value of the constant in Lemma 21 is* $c = ((p-1)^2 + \tau^2)^{\frac{p}{2}}$ *for* $2 < p < \infty$.

Proof. If $M(y) = (1+\tau^2)^{\frac{p}{2}} [y_1^2 + 2\gamma y_1 y_2 + y_2^2]^{\frac{p}{2}} + c[y_3 - (y_1 - y_2)^p]$ (where $\gamma = \frac{1-\tau^2}{1+\tau^2}$) is to be a candidate, or partial candidate, then it must agree at $y_2 = \frac{p-2}{p} y_1$, with the solution M given implicitly by the relation $G(y_1 + y_2, y_1 - y_2) = y_3 G(\sqrt{\omega^2 - \tau^2}, 1)$, from Proposition 12. At $y_2 = \frac{p-2}{p} y_1$,

$$\left(\sqrt{\omega^2 - \tau^2} + 1\right)^{p-1} \left[\sqrt{\omega^2 - \tau^2} - p + 1\right] = G\left(\sqrt{\omega^2 - \tau^2}, 1\right)$$

$$= \frac{1}{y_3}(2y_1)^{p-1}[-(p-2)y_1 + (p-2)y_1] = 0.$$

Since $\sqrt{\omega^2 - \tau^2} + 1 \neq 0$, $\sqrt{\omega^2 - \tau^2} = p - 1$, which implies $\omega = ((p-1)^2 + \tau^2)^{\frac{1}{2}}$. So,

$$\begin{aligned}
((p-1)^2 + \tau^2)^{\frac{p}{2}} y_3 &= \omega^p y_3 \\
&= M(y_1, \frac{p-2}{p} y_1, y_3) \\
&= \left[\left(2\frac{p-1}{p} y_1\right)^2 + \tau^2 \left(\frac{2}{p} y_1\right)^2\right]^{\frac{p}{2}} + c\left[y_3 - \left(\frac{2}{p} y_1\right)^p\right].
\end{aligned}$$

Now just solve for c. □

2.2.1.3 Gluing together partial candidates from Cases (1_2) and (2_2)

It turns out that the Bellman function candidate obtained from Case (2_2) is only valid on part of the domain Ξ_+, since it does not remain concave throughout (for example at or near (y_1, y_1, y_3)). As luck would have it, the partial candidate has the necessary restrictive concavity on the part of the domain where the candidate from Case (1_2) left off; i.e. in $-y_1 < y_2 < \frac{p-2}{p} y_1$. This means that we can glue together the partial candidate from Cases (1_2) and (2_2) to get a candidate on Ξ_+ for $2 < p < \infty$. The characteristics for this solution can be seen in Figure 2.5.

Proposition 23. *For* $2 < p < \infty, \gamma = \dfrac{1-\tau^2}{1+\tau^2}$ *and* $\tau \in \mathbb{R}$, *the solution to the Monge-Ampère equation is given by*

$$M(y) = (1+\tau^2)^{\frac{p}{2}}[y_1^2 + 2\gamma y_1 y_2 + y_2^2]^{\frac{p}{2}} + ((p-1)^2 + \tau^2)^{\frac{p}{2}}[y_3 - (y_1 - y_2)^p]$$

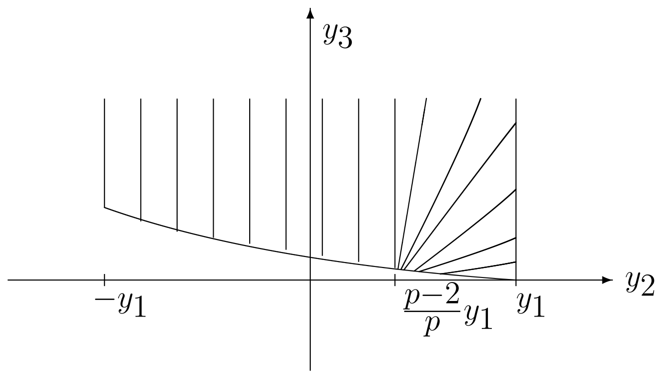

Figure 2.5: Characteristics of Bellman candidate for $2 < p < \infty$

when $-y_1 < y_2 \le \frac{p-2}{p}y_1$ *and is given implicitly by*

$G(y_1+y_2, y_1-y_2) = y_3 G(\sqrt{\omega^2-\tau^2}, 1)$ *when* $\frac{p-2}{p}y_1 \le y_2 < y_1$, *where* $G(z_1, z_2) = (z_1+z_2)^{p-1}[z_1-(p-1)z_2]$ *and* $\omega = \left(\frac{M(y)}{y_3}\right)^{\frac{1}{p}}$. *This solution satisfies all properties of the Bellman function.*

We already know that the implicit part of the solution has the correct restrictive concavity property of the Bellman function, as shown in Section 2.2.1.1. However, the restrictive concavity still needs to be verified for the explicit part. Since the explicit part of the solution satisfies $M_{y_3y_i} = M_{y_3y_3} = 0$, $D_i = 0$ for $i = 1, 2$. So all that remains to be verified for the restrictive concavity of the explicit part is checking the sign of $M_{y_iy_i}$, for $i = 1, 2$. Observe that the explicit part can be written as

$$M(y) = [(y_1+y_2)^2 + \tau^2(y_1-y_2)^2]^{\frac{p}{2}} + C_{p,\tau}[y_3 - (y_1-y_2)^p]. \tag{2.15}$$

It is easy to check that $M_{y_2y_2} \le M_{y_1y_1}$ on $-y_1 < y_2 \le \frac{p-2}{p}y_1$ for $2 < p < \infty$. So we only need to find the largest range of τ's such that $M_{y_1y_1} \le 0$.

Lemma 24. *In Case* $(2_2), M_{y_1y_1} \le 0$ *on* $-y_1 < y_2 \le \frac{p-2}{p}y_1$ *for all* $\tau \in \mathbb{R}$.

Proof. Changing coordinates back to x will make the estimates much easier. So we would like to show that, on $0 \le x_2 \le (p-1)x_1$, we have,

$$M_{y_1 y_1} \le 0, \tag{2.16}$$

where $C_{p,\tau} = ((p-1)^2 + \tau^2)^{\frac{p}{2}}$ and $\frac{1}{p} M_{y_1 y_1} =$

$$(p-2)(x_2^2 + \tau^2 x_1^2)^{\frac{p-4}{2}}(x_2 + \tau^2 x_1)^2 + (1+\tau^2)(x_2^2 + \tau^2 x_1^2)^{\frac{p-2}{2}} - (p-1)C_{p,\tau} x_1^{p-2}.$$

First, consider $4 \le p < \infty$. If $p \ne 4$, then showing (2.16) is equivalent to

$$(p-2)(p-1+\tau^2)^2 + (1+\tau^2)((p-1)^2 + \tau^2) - (p-1)((p-1)^2 + \tau^2)^2 \le 0.$$

Using the fact that $(p-1+\tau^2)^2 \le ((p-1)^2 + \tau^2)^2$, it suffices to show

$$(p-2)((p-1)^2 + \tau^2) + 1 + \tau^2 \le (p-1)((p-1)^2 + \tau^2) \iff 1 \le (p-1)^2.$$

4So we have verified that the estimate is true for all τ. Let $s = \frac{x_2}{x_1}$, then (2.16) simplifies to showing,

$$F(s) = (p-2)(s+\tau^2)^2 + (1+\tau^2)(s^2+\tau^2) - C_{p,\tau}(p-1)(s^2+\tau^2)^{\frac{4-p}{2}} \le 0,$$

where $0 \le s \le p-1$. For $p = 4, F$ is a quadratic function that is increasing on $(\dfrac{-2\tau^2}{\tau^2+3}, p-1)$. Since $F(3) \le 0$, $F(s) \le 0$ on $(0, 3)$.

Now we will consider $2 \le p < 4$. Note that $F(s) = 0$ at $p = 2$, so we can assume that

$p \neq 2$. Breaking up the domain of F will make things easier. For $s \in (1, p-1)$, we have the following estimate, $(s+\tau^2)^2 \leq (s^2+\tau^2)^2$. Let $t = s^2+\tau^2$. Then

$$\frac{1}{t}F(s) \leq (p-2)t + 1 + \tau^2 - C_{p,\tau}(p-1)t^{\frac{2-p}{2}} := g_1(t).$$

Observe that g_1 is increasing on $1+\tau^2 \leq t \leq (p-1)^2+\tau^2$ and $g_1((p-1)^2+\tau^2) \leq 0$. Therefore, $F(s) \leq 0$ on $(1, p-1)$.

Now we will show that $F(s) \leq 0$, for $s \in (0,1)$. Let $r = \tau^2 > 0$. Since $(s+r)^2 \leq (1+r)^2$, it suffices to show

$$\begin{aligned}
&(p-2)(1+r)^2 + (1+r)(s^2+r) \leq (p-1)((p-1)^2+r)^{p/2}(s^2+r)^{(4-p)/2} \\
\iff &(1+r)[s^2+(p-1)r+p-2] \leq (p-1)((p-1)^2+r)^{p/2}(s^2+r)^{(4-p)/2} \\
\iff &H(s) := \log(1+r) + \log[s^2+(p-1)r+p-2] - \log(p-1) \\
&-\frac{p}{2}\log((p-1)^2+r) - \frac{4-p}{2}\log(s^2+r) \leq 0
\end{aligned}$$

$H'(s) \geq 0$ and $H(1) \leq 0$ will imply $H \leq 0$ on $(0,1)$. $H'(s) \geq 0 \iff As^2+B \geq 0$, where $A = p-2$ and $B = 2r-(4-p)[(p-1)r+p-2]$. For this we simply need to check the conditions under which $B \geq 0$. It is easy to see that

$$B \geq 0 \iff r \geq \frac{(4-p)(p-2)}{2-(4-p)(p-1)},$$

which is satisfied for all $r \geq 0$, if $p \in [2,3]$. Checking that $H(1) \leq 0$ is easy:

$$H(1) \;=\; \frac{p}{2}\log\left(\frac{1+r}{(p-1)^2+r}\right) \leq 0$$

$$\Longleftrightarrow 1 \leq (p-1)^2,$$

which is obviously true for all $2 \leq p \leq 3$ (or better yet for all $p \geq 2$, but we don't have the derivative estimate for that large range of p's). Therefore, $F(s) \leq 0$ for $s \in (0,1)$ and $2 \leq p \leq 3$.

Let us now show that $F(s) \leq 0$, for $s \in (0,1)$ and $3 < p < 4$.

$$\begin{aligned} F(s) \;&\leq\; (p-2)(1+r)^2 + (1+r)(1+r) - (p-1)(4+r)^{p/2} r^{(4-p)/2} \leq 0 \\ &\Longleftrightarrow (1+r)^2 \leq (4+r)^{p/2} r^{(4-p)/2} \\ &\Longleftrightarrow 4\log(1+r) - 4\log r \leq p[\log(4+r) - \log r] \\ &\Longleftrightarrow p \geq \frac{4\log(1/r+1)}{\log(4/r+1)} := K(r) \end{aligned}$$

Observe that $\lim_{r\to 0^+} K(r) = 4$ and that K decreases rapidly to $\lim_{r\to\infty} K(r) = 1$. Let us find where $K(r) = 3$. This reduces to the equation

$$8r^3 + 42r^2 + 60r - 1 = 0,$$

which has a positive zero of $r \approx 0.0165$. Therefore, $K(r) \leq p$, for all $r \geq 0.02$. This gives us $F(s) \leq 0$, for $s \in (0,1), 3 < p < 4$ and $\tau^2 \geq 0.02$.

The proof is finished except that we still need to show that $F(s) \leq 0$, for $s \in (0,1)$, with $\tau^2 \leq 0.02$ and $p \in (3,4)$. We will now proceed to break $(0,1)$ into $(0, \frac{1-\tau^2}{2}) \cup (\frac{1-\tau^2}{2}, 1)$ and show that $F(s) \leq 0$ on each piece separately. Though we only need the estimate for $\tau^2 \leq 0.02$, the estimates below work for $|\tau| \leq 1$.

For $s \in (0, \frac{1-\tau^2}{2})$, we have the estimate $(s+\tau^2)^2 \leq s^2+\tau^2$. Let $t = s^2+\tau^2$. Then

$$\frac{1}{t}F(s) \leq p-1+\tau^2 - C_{p,\tau}(p-1)t^{\frac{2-p}{2}} := g_2(t).$$

Since g_2 is increasing on $\left(\tau^2, \frac{\left(1-\tau^2\right)^2}{4}+\tau^2\right)$ and since $g_2\left(\frac{(1-\tau^2)^2}{4}+\tau^2\right) \leq 0$, $F(s) \leq 0$ on $\left(0, \frac{1-\tau^2}{2}\right)$.

All that remains is to show that $F(s) \leq 0$ on $(\frac{1-\tau^2}{2}, 1)$, for $p \in (3,4)$.

$$\begin{aligned}
&\frac{1}{p-1}F(s) \leq (1+\tau^2)^2 - ((p-1)^2+\tau^2)^{\frac{p}{2}}\left(\frac{(1-\tau^2)^2}{4}+\tau^2\right)^{\frac{4-p}{2}} \leq 0 \\
&\iff 4^{\frac{4-p}{2}}(1+\tau^2)^2 \leq ((p-1)^2+\tau^2)^{\frac{p}{2}}((1-\tau^2)^2+4\tau^2)^{\frac{4-p}{2}} \\
&\iff 2^{4-p}(1+\tau^2)^{p-2} \leq ((p-1)^2+\tau^2)^{\frac{p}{2}}
\end{aligned}$$

Now we perform the second order Taylor expansion of the two terms involving p and τ and obtain

$$\begin{aligned}
&2^{4-p}[1+(p-2)\tau^2+\frac{1}{2}(p-2)(p-3)\xi_2^{p-4}\tau^4] \\
&\leq 1+\frac{p}{2}((p-1)^2+\tau^2-1)+\frac{p}{8}(p-2)((p-1)^2+\tau^2-1)^2\xi_1^{\frac{p-4}{2}},
\end{aligned}$$

where $1 \leq \xi_2 \leq 2$ and $4 \leq \xi_1 \leq 10$. Using these estimates of ξ_1 and ξ_2 and considering $p \in [3,4]$, it suffices to show

$$2+4\tau^2+\tau^4 \leq 1+\frac{3}{2}(3+\tau^2)+\frac{3}{8}(3+\tau^2)^2\frac{3}{10} \iff -71\tau^4-146\tau^2+361 \geq 0,$$

which is obviously true for all $|\tau| \leq 1$ and $p \in [3,4]$. So we have shown that $F(s) \leq 0$ on $(\frac{1-\tau^2}{2}, 1)$ for $p \in [3,4]$ and $|\tau| \leq 1$. Thus, $F(s) \leq 0$, for $s \in [0, p-1]$ and $2 < p < 4$, which completes the proof. □

We have now verified that the explicit part of the Bellman function candidate, from Case (2_2), has the appropriate restrictive concavity. So we have proven Proposition 23, by Lemmas 19, 20 and 24. Now that we have a Bellman candidate for $2 < p < \infty$, we will turn our attention to p-values in the dual range $1 < p < 2$.

2.2.2 The Bellman function candidate for $1 < p < 2$

In order to get a Bellman function candidate for $1 < p < 2$ we just need to glue together candidates from Cases (2_2) and (3_2) in almost the same way as we did for $2 < p < \infty$ in Section 2.2.1. Refer to Addendum 1 (Section 2.5) for full details.

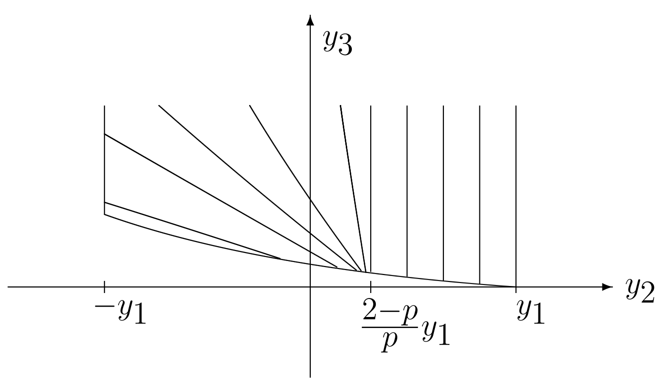

Figure 2.6: Characteristics of Bellman candidate for $1 < p < 2$.

Proposition 25. *Let $1 < p < 2$ and $\gamma = \dfrac{1-\tau^2}{1+\tau^2}$. If $\tau^2 \leq \frac{1}{2p-1}$, then a solution to the Monge-Ampère equation is given by*

$$M(y) = (1+\tau^2)^{\frac{p}{2}}[y_1^2 + 2\gamma y_1 y_2 + y_2^2]^{\frac{p}{2}} + \left(\frac{1}{(p-1)^2} + \tau^2\right)^{\frac{p}{2}} [y_3 - (y_1 - y_2)^p]$$

when $\frac{2-p}{p}y_1 \le y_2 < y_1$

and if $\tau^2 \le p^* - 1$, *then the solution is given implicitly by*

$$G(y_1 - y_2, y_1 + y_2) = y_3 G(1, \sqrt{\omega^2 - \tau^2}) \text{ when } -y_1 < y_2 \le \frac{2-p}{p}y_1, \text{ where } \omega = \left(\frac{M(y)}{y_3}\right)^{\frac{1}{p}}.$$

This solution satisfies all of the properties of the Bellman function (once we restrict the τ*-values to the smaller set between the two restrictions above).*

Most of the remaining cases do not yield a Bellman function candidate. If we fix y_2, then the Monge-Ampère solution from Cases (1) and (3) do not satisfy the restrictive concavity needed to be a Bellman function candidate. Case (2) yields the same partial solution if we first fix y_1 or y_2, since restrictive concavity is only valid on part of the domain. So, all that remains is Case (4). However, we do not know whether or not Case (4) gives a Bellman function candidate. For $\tau = 0$, it was shown in [37] that Case (4) does not produce a Bellman function candidate, since some simple extremal functions give a contradiction to linearity of the Monge-Ampère solution on characteristics. However, for $\tau \neq 0$ these extremal functions only work as a counterexample for some p-values and some signs of the martingale transform. Case (4) could give a solution throughout Ξ_+ or could yield a partial solution that would work well with the characteristics from Case (2_1). Since Case (4) does not provide a Bellman candidate for $\tau = 0$, we expect the same for small τ. The picture probably changes most drastically for large τ. But it does not matter, since we will now show that our Bellman candidate is actually the Bellman function (which we would have to check anyways because of the added assumption). The details for the remaining cases that do not yield a Bellman function candidate are in Section 2.6.

2.3 Monge-Ampère solution is the Bellman function

We will now show that the Monge-Ampère solution obtained in Proposition 23 and 25 is actually the Bellman function. To this end, let us revert back to the x-variables. We will denote the Bellman function candidate as $B_{\mathcal{T}}$ and use $\mathcal{B}_{\mathcal{T}}$ to denote the true Bellman function. Extending the function G on part of Ω_+ to $U_{\mathcal{T}}$ on all of Ω, appropriately, makes it possible to define the solution in terms of a single relation.

Definition 26. Let $v(x_1, x_2) := v_{p,\tau}(x_1, x_2) = (\tau^2|x_1|^2 + |x_2|^2)^{\frac{p}{2}} - ((p^* - 1)^2 + \tau^2)^{\frac{p}{2}}|x_1|^p$, $u(x_1, x_2) := u_{p,\tau}(x_1, x_2) = p(1 - \frac{1}{p^*})^{p-1}\left(1 + \frac{\tau^2}{(p^*-1)^2}\right)^{\frac{p-2}{2}}(|x_1| + |x_2|)^{p-1}[|x_2| - (p^* - 1)|x_1|]$ and

$$U(x_1, x_2) := U_{p,\tau}(x_1, x_2) = \begin{cases} v(x_1, x_2) & : |x_2| \geq (p^* - 1)|x_1| \\ \\ u(x_1, x_2) & : |x_2| \leq (p^* - 1)|x_1|. \end{cases}$$

for $1 < p < 2$. For $2 < p < \infty$ we interchange the two pieces in U.

Remark 27. Wherever B, U, u and v are considered, we are assuming that if $1 < p < 2$, then $\tau^2 \leq \dfrac{1}{2p - 1}$ and if $2 \leq p < \infty$, then $\tau \in \mathbb{R}$.

Proposition 28. *For $1 < p < 2$ and $\tau^2 \leq \dfrac{1}{2p-1}$ or $2 < p < \infty$ and $\tau \in \mathbb{R}$ the Bellman function candidate is the unique positive solution given by*

$$U(x_1, x_2) = U\left(x_3^{\frac{1}{p}}, \sqrt{B_{\mathcal{T}}^{\frac{2}{p}} - \tau^2 x_3^{\frac{2}{p}}}\right).$$

Furthermore, U is C^1-smooth on Ω.

Proof. First consider $2 \le p < \infty$. It is clear that

$$U(x_1, x_2) = U\left(x_3^{\frac{1}{p}}, \sqrt{B_\tau^{\frac{2}{p}} - \tau^2 x_3^{\frac{2}{p}}}\right), \tag{2.17}$$

by comparing the solution obtained in Proposition 23 and using the symmetry property in Proposition 7. The constant $\alpha_{p,\tau} = p\big(1 - \frac{1}{p^*}\big)^{p-1}\left(1 + \frac{\tau^2}{(p^*-1)^2}\right)^{\frac{p-2}{2}}$ was determined so that $U_x = U_y$ at $|y| = (p^*-1)|x|$. The partial derivatives are given by,

$$\begin{aligned}
u_{x_1} &= \alpha_{p,\tau}[(p-1)x_1'(|x_1| + |x_2|)^{p-2}(|x_2| - (p^*-1)|x_1|) - (p^*-1)x_1'(|x_1| + |x_2|)^{p-1}],\\
v_{x_1} &= p\tau^2 x_1(\tau^2|x_1|^2 + |x_2|^2)^{\frac{p-2}{2}} - px_1'((p^*-1)^2 + \tau^2)^{\frac{p}{2}}|x_1|^{p-1},\\
u_{x_2} &= \alpha_{p,\tau}(p-1)x_2'(|x_1| + |x_2|)^{p-2}(|x_2| - (p^*-1)|x_1|) + \alpha_{p,\tau}x_2'(|x_1| + |x_2|)^{p-1},\\
v_{x_2} &= px_2(\tau^2|x_1|^2 + |x_2|^2)^{\frac{p-2}{2}},
\end{aligned}$$

where $x_1' = \frac{x_1}{|x_1|}$ and $x_2' = \frac{x_2}{|x_2|}$. U is C^1-smooth, except possibly at gluing and symmetry lines. It is easy to verify that u_x is continuous at $\{x_1 = 0\}$, U_{x_1} and U_{x_2} are continuous at $\{|x_2| = (p^*-1)|x_1|\}$ and v_{x_2} is continuous at $\{x_2 = 0\}$. This proves that U is C^1-smooth on Ω.

Observe that $U_{x_2} > 0$ for $x_2 \neq 0$ and $U_{x_1} < 0$ for $x_1 \neq 0$. This is enough to show that B_τ is the unique positive solution to (2.17). Indeed, if $x \in \Omega$ such that $|x_1| = x_3^{\frac{1}{p}}$, then $\sqrt{B_\tau^{\frac{2}{p}} - \tau^2 x_3^{\frac{2}{p}}} = |x_2|$ by the Dirichlet boundary conditions. This gives us (2.17) uniquely at $B_\tau(x)$. Fix x_1, such that $|x_1| < x_3^{\frac{1}{p}}$. Then $U\left(x_3^{\frac{1}{p}}, \sqrt{B_\tau^{\frac{2}{p}} - \tau^2 x_3^{\frac{2}{p}}}\right) < U\left(x_1, \sqrt{B_\tau^{\frac{2}{p}} - \tau^2 x_3^{\frac{2}{p}}}\right)$.

Since x_1 is fixed, $\sqrt{B_\tau^{\frac{2}{p}} - \tau^2 x_3^{\frac{2}{p}}} > |x_2|$, so $U\left(x_1, \sqrt{B_\tau^{\frac{2}{p}} - \tau^2 x_3^{\frac{2}{p}}}\right)$ strictly decreases to $U(x_1, x_2)$, as $\sqrt{B_\tau^{\frac{2}{p}} - \tau^2 x_3^{\frac{2}{p}}}$ decreases to $|x_2|$, giving us a unique $B_\tau(x)$ for which (2.17) holds.

Now consider $1 < p < 2$. U is C^1-smooth on Ω, since v_{x_1} is continuous at $\{x_1 = 0\}$, u_{x_2} is continuous at $\{x_2 = 0\}$ and U_{x_1} and U_{x_2} are continuous at $\{|x_2| = (p^* - 1)|x_1|\}$. This is easily verified since the partial derivatives are computed above (just switch the two pieces of each function). Observe that for $x_1 \neq 0$ and $x_2 \neq 0, U_{x_1} < 0$ and for $x_2 \neq 0, U_{x_2} > 0$. Then the argument above showing $U(x_1, x_2) = U\left(x_3^{\frac{1}{p}}, \sqrt{B_\tau^{\frac{2}{p}} - \tau^2 x_3^{\frac{2}{p}}}\right)$ uniquely determines B_τ also holds for this range of p-values as well, except maybe at $x_1 = x_2 = 0$. Suppose $U(0,0) = U\left(x_3^{\frac{1}{p}}, \sqrt{B_\tau^{\frac{2}{p}} - \tau^2 x_3^{\frac{2}{p}}}\right)$. Then $B_\tau(x) = ((p^* - 1)^2 + \tau^2)^{\frac{p}{2}} x_3$. So $B_\tau(x)$ is uniquely determined by the fixed x-value. $\square$

Corollary 29. *B_τ is continuous in Ω, for $\tau^2 \leq \frac{1}{2p-1}$ and $1 < p < 2$ or $\tau \in \mathbb{R}$ and $2 \leq p < \infty$.*

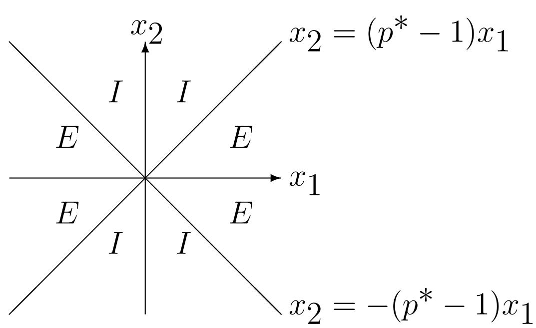

Figure 2.7: Location of Implicit (I) and Explicit (E) part of B_τ for $2 \leq p < \infty$.

Proof. In this proof only we will revert back to the notation $U_{p,\tau}$, rather than U, to make clear the distinction when $\tau = 0$ or $\tau \neq 0$. We only consider $2 < p < \infty$ as the dual range is handled identically. By Proposition 28, we have that $B_\mathcal{T}$ is the unique positive solution to (2.17). Since this is true for all $\tau \in \mathbb{R}$, $B_0 = \left(B_\mathcal{T}^{\frac{2}{p}} - \tau^2 x_3^{\frac{2}{p}}\right)^{\frac{p}{2}}$ on $|x_2| \geq (p^* - 1)|x_1|$, because $U_{p,\mathcal{T}} = \left(1 + \frac{\tau^2}{(p^*-1)^2}\right)^{\frac{p-2}{2}} U_{p,0}$. Equivalently, we have

$$B_\mathcal{T} = \left(B_0^{\frac{2}{p}} + \tau^2 x_3^{\frac{2}{p}}\right)^{\frac{p}{2}}. \tag{2.18}$$

Since B_0 was shown to be continuous in [37] (pg. 26), $B_\mathcal{T}$ is also continuous on $|x_2| \geq (p^* - 1)|x_1|$, using the relation. This takes care of the implicit part of $B_\mathcal{T}$. The explicit part of $B_\mathcal{T}$ is clearly continuous on $|x_2| \leq (p^* - 1)|x_1|$. $\square$

Lemma 30. *Let $\tau^2 \leq \frac{1}{2p-1}$ and $1 < p < 2$ or $\tau \in \mathbb{R}$ and $2 \leq p < \infty$. Then, $B_\mathcal{T}\big|_L$ is C^1-smooth on Ω, where L is any line in Ω.*

Proof. Since $B_\mathcal{T}\big|_L$ is C^2-smooth on Ω_+, all that remains to be checked is the smoothness at the gluing and symmetry lines; i.e., at $\{x_1 = 0\}, \{x_2 = 0\}$ and $\{|x_2| = (p^* - 1)|x_1|\}$. Let $L = L(t), t \in \mathbb{R}$, be any line in Ω passing through any of the planes in question, such that $L(0)$ is on the plane. Now plug $L(t)$ into (2.17) and differentiate with respect to t. Let $t \to 0^+$ and $t \to 0^-$ and equate the two relations. This gives

$$\frac{d}{dt} B_\mathcal{T}(L(t))\big|_{t=0^-} = \frac{d}{dt} B_\mathcal{T}(L(t))\big|_{t=0^+}. \qquad \square$$

Proposition 31. *(Restrictive Concavity) Let $1 < p < 2$ and $\tau^2 \leq \frac{1}{2p-1}$ or $2 \leq p < \infty$ and $\tau \in \mathbb{R}$. Suppose $x^\pm \in \Omega$ such that $x = \alpha^+ x^+ + \alpha^- x^-, \alpha^+ + \alpha^- = 1$. If $|x_1^+ - x_1^-| = |x_2^+ - x_2^-|$, then $B_\mathcal{T}(x) \geq \alpha^+ B_\mathcal{T}(x^+) + \alpha^- B_\mathcal{T}(x^-)$.*

Proof. Recall that Propositions 23 and 25, together with the symmetry property of $B_\mathcal{T}$, establish this result everywhere, except at $\{x_1 = 0\}, \{x_2 = 0\}$ and $\{|x_2| = (p^* - 1)|x_1|\}$. Let $f(t) = B_\mathcal{T}\big|_{L(t)}$, where L is any line in Ω, such that $L(0) \in \{x_1 = 0\}, \{x_2 = 0\}$ or $\{|x_2| = (p^* - 1)|x_1|\}$. Since $f'' < 0$ for $t < 0$ and $t > 0$ and f is C^1-smooth (by Lemma 30), f is concave. □

Proposition 32. *Let* $1 < p < \infty$. *If a function* $\widetilde{B}$ *has restrictive concavity and satisfies the Dirichlet estimate* $\widetilde{B}_\mathcal{T}(x_1, x_2, |x_1|^p) \geq (\tau^2 x_1^2 + x_2^2)^{\frac{p}{2}}$, *then* $\widetilde{B}_\mathcal{T} \geq \mathcal{B}_\mathcal{T}$. *In particular,* $B_\mathcal{T} \geq \mathcal{B}_\mathcal{T}$, *for* $\tau^2 \leq \frac{1}{2p-1}$ *and* $1 < p < 2$ *or* $\tau \in \mathbb{R}$ *and* $2 \leq p < \infty$.

Proof. This was proven in [37] for B_0 (Lemma 2 on page 29). The same proof will apply here to $B_\mathcal{T}$. □

Proposition 33. *For* $\tau^2 \leq \frac{1}{2p-1}$ *and* $1 < p < 2$ *or* $\tau \in \mathbb{R}$ *and* $2 \leq p < \infty, B_\mathcal{T} \leq \mathcal{B}_\mathcal{T}$.

Proof. For $1 < p < 2$ there is a direct proof, which will be discussed first. By (2.18) we know that $B_0 = \left(B_\mathcal{T}^{\frac{2}{p}} - \tau^2 x_3^{\frac{2}{p}}\right)^{\frac{p}{2}}$ on $\{|x_2| \leq (p^* - 1)|x_1|\}$. Consider, $\widetilde{\mathcal{B}}_0 = \left(\mathcal{B}_\mathcal{T}^{\frac{2}{p}} - \tau^2 x_3^{\frac{2}{p}}\right)^{\frac{p}{2}}$. It suffices to show that $B_0 \leq \widetilde{\mathcal{B}}_0$. But, $B_0 = \mathcal{B}_0$ (as Burkholder showed), so without the supremum's we can reduce to simply showing

$$\langle |g|^p \rangle_I^{\frac{2}{p}} + \tau^2 \langle |f|^p \rangle_I^{\frac{2}{p}} \leq \langle (\tau^2 |f|^2 + |g|^2)^{\frac{p}{2}} \rangle_I.$$

Apply Minkowski's inequality: $\left\| \int_I (A, C) \right\|_{l^{\frac{2}{p}}} \leq \int_I \|(A, C)\|_{l^{\frac{2}{p}}}$. Choosing $A = |g|^p$ and $C = |\tau f|^p$ proves the result. So we have shown that $B_\mathcal{T} \leq \mathcal{B}_\mathcal{T}$ on $\{|x_2| \leq (p^* - 1)|x_1|\}$.

Now we would like to show that $B_\mathcal{T} \leq \mathcal{B}_\mathcal{T}$ on $\{|x_2| \geq (p^* - 1)|x_1|\}$. Let $H_1(x_1, x_2, x_3) = B_\mathcal{T}(x_1, x_2, x_3) - B_\mathcal{T}(0, 0, 1)x_3$. Lemma 38, in the next section, proves that $H_1(x_1, x_2, \cdot)$ is an increasing function starting at $H_1(x_1, x_2, |x_1|^p) = v_\mathcal{T}(x_1, x_2)$ and increasing to $\widetilde{U}_{p,\mathcal{T}}(x, y) :=$

$\sup_{t \geq |x|^p}\{B_{\mathcal{T}}(x,y,t) - B_{\mathcal{T}}(0,0,1)t\}$. The same proof works for

$$H_2(x_1,x_2,x_3) = \mathcal{B}_{\mathcal{T}}(x_1,x_2,x_3) - \mathcal{B}_{\mathcal{T}}(0,0,1)x_3.$$

So

$$H_2(x_1,x_2,x_3) \geq v_{\mathcal{T}}(x_1,x_2) = B_{\mathcal{T}}(x_1,x_2,x_3) - B_{\mathcal{T}}(0,0,1)x_3.$$

Since $B_{\mathcal{T}}(0,0,1) \leq \mathcal{B}_{\mathcal{T}}(0,0,1)$, $B_{\mathcal{T}} \leq \mathcal{B}_{\mathcal{T}}$ on $\{|x_2| \geq (p^*-1)|x_1|\}$.

Now we consider $2 < p < \infty$. Let $\varepsilon > 0$ be arbitrarily small and consider the following extremal functions

$$f(x) = \begin{cases} -c & : 1 < x < \varepsilon \\ \gamma f\left(\frac{t-\varepsilon}{1-2\varepsilon}\right) & : \varepsilon < x < 1-\varepsilon \\ c & : 1-\varepsilon < x < 1, \end{cases}$$

$$g(x) = \begin{cases} d_- & : 1 < x < \varepsilon \\ \gamma g\left(\frac{t-\varepsilon}{1-2\varepsilon}\right) & : \varepsilon < x < 1-\varepsilon \\ d_+ & : 1-\varepsilon < x < 1, \end{cases}$$

where $c, d_\pm$ and γ are defined so that f and g are a pair of test functions at $(0,x_2,x_3)$. We can use f and g to show, just as in [37] (Lemma 3, pg. 30), that

$$B_{\mathcal{T}}(0,x_2,x_3) \leq \mathcal{B}_{\mathcal{T}}(0,x_2,x_3). \tag{2.19}$$

Now we need to take care of the estimate when $x_1 \neq 0$. Making a change of coordinates from x to y we only need to consider $y \in \Xi_+$, by the symmetry property of the Bellman function and Bellman function candidate. So far we have that $M_{\mathcal{T}}(y_1,y_1,y_3) \leq$

$\mathcal{M}_\mathcal{T}(y_1, y_1, y_3)$ by (2.19). The Dirichlet boundary conditions give that $M(y_1, y_2, (y_1 - y_2)^p) = \mathcal{M}(y_1, y_2, (y_1 - y_2)^p)$. On any characteristic in $\{\frac{p-2}{p}y_1 \leq y_2 \leq y_1\}$, see Figure 2.5, $M_\mathcal{T}$ is linear (since it is the Monge-Ampère solution) and $\mathcal{M}_\mathcal{T}$ is concave (by Proposition 5). Therefore, $M_\mathcal{T}(y_1, y_2, y_3) \leq \mathcal{M}_\mathcal{T}(y_1, y_2, y_3)$ on $\{\frac{p-2}{p}y_1 \leq y_2 \leq y_1\}$. For the remaining part of Ξ_+, we can use the same proof as for $1 < p < 2$ to get $M_\mathcal{T}(y_1, y_2, y_3) \leq \mathcal{M}_\mathcal{T}(y_1, y_2, y_3)$ on $\{-y_1 \leq! y_2 \leq \frac{p-2}{p}y_1\}$. □

Now that we have proven $B = \mathcal{B}$, we will mention another surprising fact.

Definition 34. We define $\mathcal{B}^l = \mathcal{B}^l(x_1, x_2, x_3)$ as the least restrictively concave majorant of $(x_2^2 + \tau^2 x_1^2)^{\frac{p}{2}}$ in Ω.

Proposition 35. *For $1 < p < 2$ and $\tau^2 \leq \frac{1}{2p-1}$ or $2 \leq p < \infty$ and $\tau \in \mathbb{R}$ we have $B = \mathcal{B} = \mathcal{B}^l$.*

Proof. First we will show $\mathcal{B} \geq \mathcal{B}^l$. By Minkowski's inequality,

$$
\begin{aligned}
B(x_1, x_2, x_3) = \mathcal{B}(x_1, x_2, x_3) &\geq \int (|g|^2 + \tau^2 |f|^2)^{\frac{p}{2}} = \int \|(g, \tau f)\|_{l^2}^{p} \geq \left\| \int (g, \tau f) \right\|_{l^2}^{p} \\
&= (|\langle g \rangle|^2 + \tau^2 |\langle f \rangle|^2)^{\frac{p}{2}} = (x_2^2 + \tau^2 x_1^2)^{\frac{p}{2}},
\end{aligned}
$$

proving the estimate.

Conversely, one can show that $\mathcal{B} \leq \mathcal{B}^l$ just as $\mathcal{B} \leq B$ was shown in Proposition 32 (simply apply the same argument to $\mathcal{B}^l$ using the restrictive concavity of this function). □

2.4 Proving the main result

Now that we have the Bellman function, the main result can be proven without too much difficulty. But first, we will find another relationship between U and v. Quite surprisingly, U is the least zigzag-biconcave majorant of v.

Definition 36. A function of (x, y) that is biconcave in $(x+y, x-y)$ we call *zigzag-biconcave.*

Remark 37. For this section, while proving the main result, we will stop using (x_1, x_2, x_3) and (y_1, y_2, y_3) coordinates to denote a rotation by $\frac{\pi}{4}$. Definition 36 makes the rotation clear and we can simply use (x, y, t) as the coordinates and relax the use of subscripts.

Lemma 38. *Let* $1 < p < \infty$ *and* $\widetilde{U}(x, y) = \sup_{t \geq |x|^p} \{B_\mathcal{T}(x, y, t) - B_\mathcal{T}(0, 0, 1)t\}$. *Fix* (x, y). *The function* $H(x, y, t) = B_\mathcal{T}(x, y, t) - B_\mathcal{T}(0, 0, 1)t$ *is increasing in* t *from* $H(x, y, |x|^p) = v(x, y) := (\tau^2|x|^2 + |y|^2)^{\frac{p}{2}} - ((p^* - 1)^2 + \tau^2)^{\frac{p}{2}}|x|^p$ *to* $\widetilde{U}_{p,\mathcal{T}}(x, y)$.

Proof. Recall that $B_\mathcal{T}$ is continuous in Ω and for (x, y) fixed, $B_\mathcal{T}(x, y, \cdot)$ is concave. Then $H(x, y, \cdot)$ is also concave. Since $\widetilde{U}_{p,\mathcal{T}}(x, y) = \sup_{t \geq |x|^p} \{B_\mathcal{T}(x, y, t) - B_\mathcal{T}(0, 0, 1)t\}$, it either increases to $\widetilde{U}(x, y)$, or there exists t_0 such that $H(x, y, t_0) = \widetilde{U}(x, y)$ and H is decreasing for $t > t_0$. If H is decreasing for $t > t_0$, then $H \to -\infty$ as $t \to \infty$ by concavity. Then there exists $\varepsilon > 0$ and $t' > t_0$ such that $H(x, y, t') < \varepsilon t'$. So we have, $\limsup_{t \to \infty} \frac{H(x,y,t)}{t} < -\varepsilon$. But,

$$\lim_{t \to \infty} \frac{H(x, y, t)}{t} = \lim_{t \to \infty} \left[B_\mathcal{T} \left(\frac{x}{t^{\frac{1}{p}}}, \frac{y}{t^{\frac{1}{p}}}, 1 \right) - B_\mathcal{T}(0, 0, 1) \right] = 0,$$

by continuity of $B_\mathcal{T}$ at $(0, 0, 1)$. This gives us a contradiction. Therefore, $H(x, y, t) \geq -\varepsilon t$, for all t and all $\varepsilon > 0$; i.e., H is non-negative concave function on $[|x|^p, \infty)$. So $H(x, y, \cdot)$ is increasing and $H(x, y, |x|^p) = v_{p,\mathcal{T}}(x, y)$ by the Dirichlet boundary conditions of $B_\mathcal{T}$ in Proposition 7. $\square$

Proposition 39. *For $1 < p < 2$ and $\tau^2 \leq \frac{1}{2p-1}$ or $2 \leq p < \infty$ and $\tau \in \mathbb{R}, U_{p,\tau}(x, y) = \widetilde{U}_{p,\mathcal{T}}(x, y)$.*

Proof. Suppose $2 \leq p < \infty$ and $|y| \geq (p-1)|x|$. Then

$$\begin{aligned}
\widetilde{U}_0(x,y) &= \lim_{t\to\infty} (B_0(x,y,t) - B_0(0,0,1)t) \\
&= \lim_{t\to\infty} \frac{B_0\left(\frac{x}{t^{1/p}}, \frac{y}{t^{1/p}}, 1\right) - B_0(0,0,1)}{1/t} \\
&= \frac{d}{du} B_0(u^{1/p}x, u^{1/p}y, 1)\bigg|_{u=0}.
\end{aligned}$$

Now we repeat the same steps and obtain

$$\begin{aligned}
\widetilde{U}_{\mathcal{T}}(x,y) &= \lim_{t\to\infty} (B_{\mathcal{T}}(x,y,t) - B_{\mathcal{T}}(0,0,1)t) \\
&= \frac{d}{du}\left[\left(B_0^{2/p}(u^{1/p}x, u^{1/p}y, 1) + \tau^2\right)^{p/2}\right]\bigg|_{u=0} \\
&= \left[\left(B_0^{\frac{2}{p}}(u^{\frac{1}{p}}x, u^{\frac{1}{p}}y, 1) + \tau^2\right)^{\frac{p-2}{2}} B_0^{\frac{2-p}{p}}(u^{\frac{1}{p}}x, u^{\frac{1}{p}}y, 1)\frac{d}{du}B_0(u^{\frac{1}{p}}x, u^{\frac{1}{p}}y, 1)\right]\bigg|_{u=0} \\
&= \left(1 + \frac{\tau^2}{(p-1)^2}\right)^{\frac{p-2}{2}} \widetilde{U}_0(x,y) = \left(1 + \frac{\tau^2}{(p-1)^2}\right)^{\frac{p-2}{2}} U_0(x,y),
\end{aligned}$$

where the last equality is by [15]. Therefore, $\widetilde{U}_{\mathcal{T}}(x,y) = U_{\mathcal{T}}(x,y)$.

Now suppose $|y| \leq (p-1)|x|$. Looking at the explicit form of $B_{\mathcal{T}}$ in the region, note that $B_{\mathcal{T}}(x, y, \cdot)$ is linear. So

$$\widetilde{U}_{\mathcal{T}}(x,y) = \sup_{t \geq |x|^p} \{B_{\mathcal{T}}(x,y,t) - B_{\mathcal{T}}(0,0,1)t\}$$

$$= \sup_{t \geq |x|^p} \{B_{\mathcal{T}}(x,y,0)\} = v_{\mathcal{T}}(x,y) = U_{\mathcal{T}}(x,y).$$

We can apply the same proof to show that $\widetilde{U}_{\mathcal{T}}(x,y) = U_{\mathcal{T}}(x,y)$ for $1 < p < 2$. $\square$

Proposition 40. *For $\tau^2 \leq \frac{1}{2p-1}$ and $1 < p < 2$ or $\tau \in \mathbb{R}$ and $2 \leq p < \infty$, U is the least zigzag-biconcave majorant of $v(x,y) = (y^2 + \tau^2 x^2)^{\frac{p}{2}} - ((p^*-1)^2 + \tau^2)^{\frac{p}{2}} |x|^p$.*

Proof. Recall the following facts just proved in Lemma 38 and Proposition 39:

$$U(x,y) = \sup_{t:(x,y,t)\in\Omega} \{\mathcal{B}(x,y,t) - \mathcal{B}(0,0,1)t\} \geq v(x,y) = (y^2 + \tau^2 x^2)^{\frac{p}{2}} - C_p t,$$

where $C_p = ((p^*-1)^2 + \tau^2)^{\frac{p}{2}}$.

Suppose that w is a zigzag-biconcave function such that $v \leq w \leq U$. Then $W(x,y,t) := w(x,y) + C_p t$ has restrictive concavity and $W(x,y,t) \geq v(x,y) + C_p t = (y^2 + \tau^2 x^2)^{\frac{p}{2}}$. Therefore, by Proposition 32, we have $W \geq \mathcal{B}$. So we have

$$\begin{aligned} w(x,y) &= \sup_{t:(x,y,t)\in\Omega} \{W(x,y,t) - C_p t\} \\ &\geq \sup_{t:(x,y,t)\in\Omega} \{\mathcal{B}(x,y,t) - C_p t\} = U(x,y). \end{aligned}$$ $\square$

We now have enough machinery to easily prove the main result, in terms of the Haar expansion of a $\mathbb{R}$-valued L^p function.

Theorem 41. *Let $1 < p < 2$ and $\tau^2 \leq \frac{1}{2p-1}$ or $2 \leq p < \infty$ and $\tau \in \mathbb{R}$. Let $f, g : [0,1] \to \mathbb{R}$. If $|\langle g \rangle_{[0,1]}| \leq (p^*-1)|\langle f \rangle_{[0,1]}|$ and $|(f, h_J)| = |(g, h_J)|$ for all $J \in \mathcal{D}$, then $\big\langle (\tau^2|f|^2 + |g|^2)^{\frac{p}{2}} \big\rangle_{[0,1]} \leq \big((p^*-1)^2 + \tau^2\big)^{\frac{p}{2}} \langle |f|^p \rangle_{[0,1]}$, where $((p^*-1)^2 + \tau^2)$ is the sharp constant and $p^* - 1 = \max\big\{p-1, \frac{1}{p-1}\big\}$.*

Proof. Suppose that $2 \leq p < \infty$ and $\tau \in \mathbb{R}$. The proof relies on the fact that $B = \mathcal{B}$ (Propositions 39 and 33) and $U(x,y) = \sup_{t \geq |x|^p}\{B(x,y,t) - B(0,0,1)t\}$ (Lemma 39 and Proposition 38).

Since $|y| \leq (p^* - 1)|x|$ on Ω,

$$U(x,y) = v(x,y) = (|y|^2 + \tau^2|x|^2)^{\frac{p}{2}} - ((p^* - 1)^2 + \tau^2)^{\frac{p}{2}}|x|^p \leq 0.$$

Then,

$$\sup_{\substack{t>|x|^p \\ |y|\leq(p^*-1)|x|}} \{B(x,y,t) - B(0,0,1)t\} \leq 0.$$

But, $U(0,0) = 0$. Therefore

$$\sup_{\substack{t>|x|^p \\ |y|\leq(p^*-1)|x|}} \frac{B(x,y,t)}{t} = B(0,0,1) = ((p^* - 1)^2 + \tau^2)^{\frac{p}{2}}. \tag{2.20}$$

Observing the relationship $B = \mathcal{B}$, gives the desired result.

For $1 < p < 2$, $\tau^2 \leq \frac{1}{2p-1}$ and $|y| \leq (p^* - 1)|x|$,

$$U(x,y) = p\left(1 - \frac{1}{p^*}\right)\left(1 + \frac{\tau^2}{(p^* - 1)^2}\right)^{\frac{p-2}{2}} (|x| + |y|)^{p-1}[|y| - (p^* - 1)|x|] \leq 0,$$

so we have (2.20) by the same reasoning as for $2 \leq p < \infty$. $\square$

Remark 42. Note that Minkowski's inequality together with Burkholder's original result gives the same upper estimate for $2 \leq p < \infty$.

Indeed, if $f \in L^p[0,1]$ and g is the corresponding martingale transform, then Minkowski's inequality gives,

$$\|g^2+\tau^2 f^2\|_{L^{\frac{p}{2}}}^{\frac{p}{2}} \le (\|g^2\|_{L^{\frac{p}{2}}}+\|\tau^2 f^2\|_{L^{\frac{p}{2}}})^{\frac{p}{2}} = (\|g\|_{L^p}^2+\|\tau f\|_{L^p}^2)^{\frac{p}{2}}$$
$$\le \|f\|_{L^p}^p((p^*-1)^2+\tau^2)^{\frac{p}{2}}.$$

This is very surprising in the sense that the "trivial" constant $\big((p^*-1)^2+\tau^2\big)^{\frac{p}{2}}$ is actually the sharp constant.

Now we will prove the main result for Hilbert-valued martingales. The same ideas can be used to extend the previous result to Hilbert-valued L^p-functions as well. Let $\mathbb{H}$ be a separable Hilbert space with $\|\cdot\|_{\mathbb{H}}$ as the induced norm.

Theorem 43. *Let $1<p<\infty, (W,\mathcal{F},\mathbb{P})$ be a probability space and $\{f_k\}_{k\in\mathbb{Z}^+}, \{g_k\}_{k\in\mathbb{Z}^+} : W\to\mathbb{H}$ be two $\mathbb{H}$-valued martingales with the same filtration $\{\mathcal{F}_k\}_{k\in\mathbb{Z}^+}$. Denote $d_k = f_k-f_{k-1}, d_0=f_0, e_k=g_k-g_{k-1}, e_0=g_0$ as the associated martingale differences. If $\|e_k(\omega)\|_{\mathbb{H}} \le \|d_k(\omega)\|_{\mathbb{H}}$, for all $\omega\in W$ and all $k\ge 0$, then we have the following estimate*

$$\Big\|\Big(\sum_{k=0}^n e_k, \tau\sum_{k=0}^n d_k\Big)\Big\|_{L^p(W,\mathbb{H}^2)} \le ((p^*-1)^2+\tau^2)^{\frac{p}{2}}\Big\|\sum_{k=0}^n d_k\Big\|_{L^p(W,\mathbb{H})},$$

with $\big((p^*-1)^2+\tau^2\big)^{\frac{p}{2}}$ as the best possible constant for and $\tau^2\le\frac{1}{2p-1}$ and $1<p<2$ or $\tau\in\mathbb{R}$ and $2\le p<\infty$, where $p^*-1=\max\{p-1,\frac{1}{p-1}\}$.

In the theorem, "best possible" constant means that if $C_{p,\tau} < ((p^*-1)^2+\tau^2)^{\frac{1}{2}}$, then for some probability space $(W,\mathcal{G},P)$ and a filtration $\mathcal{F}$, there exists $\mathbb{H}$-valued martingales

$\{f\}_k$ and $\{g\}_k$, such that

$$\left\|(g_k, \tau f_k)\right\|_{L^p([0,1],\mathbb{H}^2)} > C_{p,\tau} \left\|f_k\right\|_{L^p([0,1],\mathbb{H})}.$$

Proof. We will prove the result for $2 \leq p < \infty$, since the result for $1 < p < 2$ is similar. Replace $|\cdot|$ with $\|\cdot\|_{\mathbb{H}}$, in $U_{p,\tau}$. Let $f_n = \sum_{k=0}^{n} d_k$ and $g_n = \sum_{k=0}^{n} e_k$. Recall that $U := U_{p,\tau}$ is the least zigzag-biconcave majorant of $v := v_{p,\tau}$. As in [16] (pages 77-79),

$$U_{p,\tau}(x+h, y+k) \leq U_{p,\tau}(x,y) + \Re(\partial_x U_{p,\tau}, h) + \Re(\partial_y U_{p,\tau}, k), \tag{2.21}$$

for all $x, y, h, k \in \mathbb{H}$, such that $|k| \leq |h|$ and $\|x + ht\|_{\mathbb{H}}\|x + kt\|_{\mathbb{H}} > 0$. The result in (2.21) follows from the zigzag-biconcavity and implies that $\mathbb{E}[U(f_k, g_k)]$ is a supermartingale. Lemma 38 gives that $v(f_n, g_n) \leq U(f_n, g_n)$. Therefore,

$$\mathbb{E}[v(f_n, g_n)] \leq \mathbb{E}[U(f_n, g_n)] \leq \mathbb{E}[U(f_{n-1}, g_{n-1})] \leq \cdots \leq \mathbb{E}[U(d_0, e_0)].$$

But, $\mathbb{E}[U(d_0, e_0)] \leq 0$ in both pieces of U_τ since $2 - p^* \leq 0$ and $\|e_0\|_{\mathbb{H}} \leq \|d_0\|_{\mathbb{H}}$. Thus, $\mathbb{E}[v_\tau(f_n, g_n)] \leq 0$. The constant, in the estimate, is best possible, since it was attained in Theorem 41. $\square$

The following construction originated from a conversation with Fedja Nazarov.

Remark 44. For $1 < p < 2$ and $\tau^2 > \frac{1}{2p-1}$, chosen sufficiently large, the "trivial" constant, $\left((p^*-1)^2 + \tau^2\right)^{\frac{p}{2}}$, in the main result is no longer sharp because of a "phase transition". To give a sense of why this is true one can show that for $1 < p < 2$ fixed, the constant is no longer sharp for τ sufficiently large.

Let us construct such a function f to do this. First of all, $f_n \in L^p[0,1]$ will be chosen so that $f_n \neq 0$ a.e. Let $C_p = (p^* - 1)$. Note that

$$\begin{aligned}
\int (|g_n|^2 + \tau^2 |f_n|^2)^{\frac{p}{2}} &= |\tau|^p \int |f_n|^p \left(1 + \frac{1}{|\tau|^2} \frac{|g_n|^2}{|f_n|^2}\right)^{\frac{p}{2}} \\
&= |\tau|^p \int |f_n|^p + \frac{p}{2|\tau|^{2-p}} \int \frac{|g_n|^2}{|f_n|^{2-p}} \\
&\quad + \frac{p}{2}\left(\frac{p}{2} - 1\right) \frac{1}{2|\tau|^{4-p}} \int \frac{|g_n|^4}{|f_n|^{4-p}} \left(\frac{1}{1+\theta_n(x,\tau)}\right)^{2-\frac{p}{2}} \\
&=: |\tau|^p \int |f_n|^p + A + B
\end{aligned}$$

$$\begin{aligned}
((p^*-1)^2 + \tau^2)^{\frac{p}{2}} \int |f_n|^p &= |\tau|^p \left(1 + \frac{C_p^2}{|\tau|^2}\right)^{\frac{p}{2}} \int |f_n|^p \\
&= |\tau|^p \int |f_n|^p + \frac{pC_p^2}{2|\tau|^{2-p}} \int |f_n|^p \\
&\quad + \frac{p}{2}\left(\frac{p}{2} - 1\right) \frac{C_p^4}{2|\tau|^{4-p}} \int |f_n|^p \left(\frac{1}{1+\tilde{\theta}_n}\right)^{2-\frac{p}{2}} \\
&=: |\tau|^p \int |f_n|^p + C + D,
\end{aligned}$$

where $\theta_n(x,\tau), \tilde{\theta}_n \geq 0$.

Choose $f_n = \chi_{[\frac{1}{8},\frac{1}{4})\cup[\frac{5}{8},\frac{3}{4})} - \chi_{[\frac{3}{8},\frac{1}{2})\cup[\frac{7}{8},1)} - \varepsilon_n \chi_{[0,\frac{1}{8})\cup[\frac{1}{2},\frac{5}{8})} + \varepsilon_n \chi_{[\frac{1}{4},\frac{3}{8})\cup[\frac{3}{4},\frac{7}{8})}$, where $\varepsilon_n > 0$ is small. On the sets where $|f_n| = \varepsilon_n$, we can choose the martingale transform g_n of f_n not small. Indeed, without loss of generality choose $x \in \big[0, \frac{1}{8}\big)$ and denote $J_1 = \big[0, \frac{1}{4}\big)$. Then

$$f_n(x) = \sum_{I : I \supset J_1} (f, h_I) h_I(x) = -\varepsilon_n.$$

Define a martingale transform g_n as

$$g_n(x) = \sum_{I: I \supsetneq J_1} (f, h_I) h_I(x) - (f, h_{J_1}) h_{J_1}(x).$$

Then

$$f_n(x) - g_n(x) = 2(f_n, h_{J_1}) h_{J_1}(x) = (2 - \varepsilon_n)\sqrt{|J_1|}\Big(-\frac{1}{\sqrt{|J_1|}}\Big).$$

Therefore, $f_n(x) = -\varepsilon_n$, yet its martingale transform $g_n(x) = 2 - 2\varepsilon_n$, for $x \in J_1$. The same can be done for other intervals of smallness of f_n. Note that $\int |g_n|^2 = \int |f_n|^2 \to \int |f|^2 = \frac{1}{2}$, if we choose $\varepsilon_n \to 0$. So $\int |g_n|^2 \approx \frac{1}{2}$, for n sufficiently large.

To show that $\int (g^2 + \tau^2 f^2)^{\frac{p}{2}} > ((p^* - 1)^2 + \tau^2)^{\frac{p}{2}} \int |f|^p$ it suffices to show $A + B > C + D$. As $D \leq 0$ it is enough to prove $A + B - C > 0$. Let $A' = \tau^{2-p} A$, $C' = \tau^{2-p} C$, $B' = \tau^{4-p} B$. Regardless of the choice of τ we have $A' > 2C'$ if n is chosen sufficiently large. In fact looking at A' we see that it is bigger than the integral, where the integrand has numerator close to 2 and denominator equal ε_n. On the other hand C' involves just an integral with uniformly (in n) bounded integrand. Then we fix n, of course $|B'|$ is very large, but we notice that choosing τ to be very large makes the following inequality true:

$$A + B - C = \frac{1}{|\tau|^{2-p}}[(A' - C') - \frac{1}{|\tau|^2}|B'|] > 0\,.$$

This completes the example.

Remark 45. We would like to point out that that as p approaches 2 from the left that the τ values for the main result do not improve to infinity, as we expect. So either there can be some improvement in the range of τ-values for which we have the Bellman function, or the operator which we are studying in this chapter behaves in a very nonintuitive way.

2.5 Proof of Proposition 25

Throughout this section the arguments may seem brief in comparison to Section 2.2.1. The reason for this is because we cover the exact same argument as in Section 2.2.1, only with slightly different cases. So if any arguments are unclear, then returning to Section 2.2.1 should help to clear up any difficulties. We will first consider Case (3_2) to get a partial Bellman function candidate.

2.5.1 Considering Case (3_2)

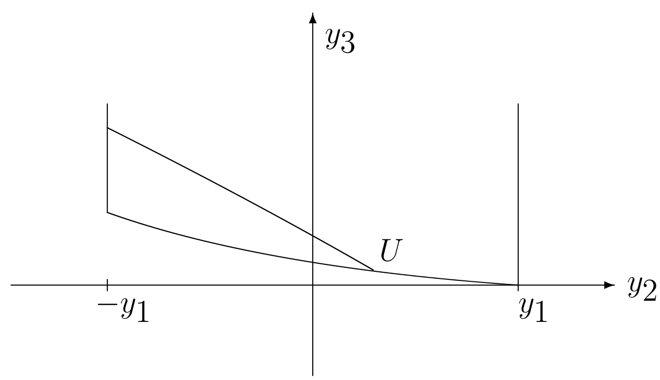

Figure 2.8: Sample characteristic of Monge-Ampère solution in Case (3_2)

Proposition 46. *For $1 < p < \infty$ and $-y_1 < y_2 < \frac{2-p}{p}y_1$, M is given implicitly by the relation $G(y_1 - y_2, y_1 + y_2) = y_3 G(1, \sqrt{\omega^2 - \tau^2})$.*

This is proven through a series of Lemmas.

Lemma 47. *$M(y) = t_2y_2 + t_3y_3 + t_0$ on the characteristic $y_2dt_2 + y_3dt_3 + dt_0 = 0$ can be simplified to $M(y) = \left(\frac{\sqrt{(y_1+u)^2+\tau^2(y_1-u)^2}}{y_1-u}\right)^p y_3$, where u is the unique solution to the equation $\frac{y_2+(1-\frac{2}{p})y_1}{y_3} = \frac{u+(1-\frac{2}{p})y_1}{(y_1-u)^p}$ and $-y_1 < y_2 < \frac{2-p}{p}y_1$.*

Proof. Any characteristic, in Case(3_2), goes from $U = \big(y_1, u, (y_1-u)^p\big)$ to $W = (y_1, -y_1, w)$. Recall the properties of the Bellman function we derived in Proposition 7, as we will be using them throughout the proof. Using the Neumann property and the property from Proposition 9, we get $M_{y_1} = -M_{y_2} = -t_2$ at W. By homogeneity at W we get

$$-py_1t_2 + pwt_3 + pt_0 = pM(W) = y_1M_{y_1} + y_2M_{y_2} + py_3M_{y_3} = -2y_1t_2 + pwt_3.$$

Now we follow the same idea as in Lemma 13, to get

$$M(y) = \left(\frac{\sqrt{(y_1+u)^2+\tau^2(y_1-u)^2}}{y_1-u}\right)^p y_3,$$

where $u = u(y_1, y_2, y_3)$ is the solution to the equation

$$\frac{y_2 + (1-\frac{2}{p})y_1}{y_3} = \frac{u + (1-\frac{2}{p})y_1}{(y_1-u)^p}. \tag{2.22}$$

Fix $u = -(1-\frac{2}{p})y_1$. Then we see that $y_2 = -(\frac{2}{p}-1)y_1 = u$ is also fixed by (2.22). This means that the characteristics must lie in the sector shown in Figure 2.9, since they go from U to $W \in \{y_2 = -y_1\}$. The same argument as in Lemma 13 can be used to verify that equation (2.22) has a unique solution in the sector $-y_1 < y_2 < \frac{2-p}{p}y_1$. $\square$

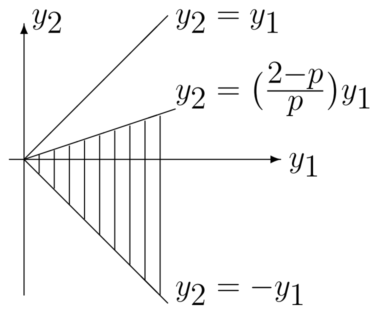

Figure 2.9: Range of characteristics in Case (3_2) for $1 < p < 2$.

Lemma 48. $M(y) = \left(\frac{\sqrt{(y_1+u)^2+\tau^2(y_1-u)^2}}{y_1-u}\right)^p y_3$ *can be rewritten as*

$$G(y_1 - y_2, y_1 + y_2) = y_3 G(1, \sqrt{\omega^2 - \tau^2}), \ \textit{for} \ -y_1 < y_2 < \frac{2-p}{p} y_1.$$

Proof. $\omega = \left(\frac{M(y)}{y_3}\right)^{\frac{1}{p}} = \frac{\sqrt{(y_1+u)^2+\tau^2(y_1-u)^2}}{y_1-u} \geq |\tau|$

Since $y_1 \pm u \geq 0$ and since$\omega^2 - \tau^2 \geq 0$, $u = \frac{\sqrt{\omega^2-\tau^2}-1}{\sqrt{\omega^2-\tau^2}+1} y_1$ by inversion. Substituting u into $\frac{y_2+(1-\frac{2}{p})y_1}{y_3} = \frac{u+(1-\frac{2}{p})y_1}{(y_1-u)^p}$ gives

$$2^{p-1} y_1^{p-1}[py_2 + (p-2)y_1] = y_3(\sqrt{\omega^2 - \tau^2} + 1)^{p-1}[\sqrt{\omega^2 - \tau^2} - (p-1)]$$

or $(x_1 + x_2)^{p-1}[(p-1)x_2 - x_1]$

$$= \left[\sqrt{B^{2/p} - \left(\tau x_3^{1/p}\right)^2} + x_3^{1/p}\right]^{p-1} \left[(p-1)\sqrt{B^{2/p} - \left(\tau x_3^{1/p}\right)^2} - x_3^{1/p}\right].$$

Thus, $G(x_1, x_2) = G\left(x_3^{1/p}, \sqrt{B^{2/p} - \left(\tau x_3^{1/p}\right)^2}\right)$ or equivalently

$$G(y_1 - y_2, y_1 + y_2) = y_3 G(1, \sqrt{\omega^2 - \tau^2}). \qquad \square$$

As before, we must verify that this partial Bellman function candidate has the restrictive concavity property, so y_1 is no longer fixed. To check restrictive concavity, we must show that $M_{y_1y_1} \le 0, M_{y_2y_2} \le 0, M_{y_3y_3} \le 0$ and $D_1 \ge 0$ (note that $D_2 = 0$ by assumption). These estimates are verified in the following series of lemmas.

Lemma 49. *In Case* (3_2) *we choose* $H(y_1, y_2) = G(y_1 - y_2, y_1 + y_2)$ *because of how the implicit solution is defined and obtain* $\operatorname{sign} H'' = -\operatorname{sign}(p-2)$.

Proof. We already computed

$$H'' = \begin{cases} 4G_{z_1z_2}, & \alpha_j = \beta_j \\ 0, & \alpha_j = -\beta_j \end{cases}$$

in Lemma 16. Since, $\alpha_1 = 1, \alpha_2 = -1, \beta_1 = 1$ and $\beta_2 = 1$ then $G_{z_1z_2} = -p(p-1)(p-2)(y_1 + y_2)(2y_1)^{p-3}$. □

Remark 50. In Case $(3_2), \beta > \frac{1}{p-1}$ in the sector $-y_1 < y_2 < \frac{2-p}{p} y_1$, where $\beta := \sqrt{\omega^2 - \tau^2}$. Equivalently, $B(x_1, x_2, x_3) \ge ((p^* - 1)^2 + \tau^2)^{\frac{p}{2}} x_3$ in $-y_1 < y_2 < \frac{2-p}{p} y_1$.

This is trivial since

$$\begin{aligned}(\beta + 1)^{p-1}[1 - (p-1)\beta] = G(1, \beta) &= \frac{1}{y_3} G(y_1 - y_2, y_1 + y_2) \\ &= (2y_1)^{p-1}[(p-2)y_1 + py_2] < 0.\end{aligned}$$

Now we have enough information to check the sign of D_1. We will start limiting the values of τ, since it will be essential for having the restrictive concavity of the partial Bellman candidate from Case (2_2).

Lemma 51. *$D_1 > 0$ in Case (3_2) for all τ-values such that $\frac{1}{p-1} \geq \tau^2$.*

Proof. We use the partial derivatives of G computed in the proof of Lemma 16 to make the computations of Φ' and Φ'' easier.

$$\begin{aligned}
\Phi(\omega) &= G(1, \beta) \\
\Phi'(\omega) &= -p(p-1)\omega[\beta+1]^{p-2} \qquad (2.23) \\
\Phi''(\omega) &= -\frac{p(p-1)(1+\beta)^{p-3}}{\beta}\left[\beta(1+\beta) + (p-2)\omega^2\right]
\end{aligned}$$

$$\begin{aligned}
\Lambda &= (p-1)\Phi' - \omega\Phi'' \\
&= -p(p-1)^2\omega(\beta+1)^{p-2} + \frac{p(p-1)\omega(1+\beta)^{p-3}}{\beta}\left[\beta(1+\beta) + (p-2)\omega^2\right] \\
&= \frac{p(p-1)\omega(1+\beta)^{p-3}}{\beta}\left[-(p-1)(1+\beta)\beta + \beta(1+\beta) + (p-2)\omega^2\right] \\
&= -\frac{p(p-1)(p-2)\omega(1+\beta)^{p-3}[\beta-\tau^2]}{\beta} \qquad (2.24)
\end{aligned}$$

Thus, $\operatorname{sign} D_1 = \operatorname{sign} H'' \operatorname{sign} \Lambda = [-\operatorname{sign}(p-2)]^2 \operatorname{sign}(\beta - \tau^2)$ by (2.12) and Lemma 49. In order to have $D_1 > 0$, we must have that $\beta > \tau^2$. By Remark 50, $\beta > \frac{1}{p-1}$. So if we impose that $\frac{1}{p-1} \geq \tau^2$ then we will have $\beta > \tau^2$ and therefore $D_1 > 0$. $\square$

The following lemma restricts the p-values for which our solution is a Bellman function candidate to $1 < p < 2$.

Lemma 52. $\operatorname{sign} M_{y_1y_1} = \operatorname{sign} M_{y_2y_2} = \operatorname{sign} M_{y_3y_3} = \operatorname{sign}(p-2)$ *in Case (3_2) for all τ such that $\tau^2 \leq p^* - 1$. Consequently, M is a Bellman function candidate for $1 < p < 2$ but not for $2 < p < \infty$, since it would not satisfy the restrictive concavity needed.*

Proof. By (2.10), (2.23), (2.24)),

$$M_{y_3 y_3} = \frac{p\omega^{p-2} R_1^2 H^2}{y_3^3} \left[\frac{\Lambda}{\Phi'}\right],$$

giving sign $M_{y_3 y_3} = (-1)[-\operatorname{sign}(p-2)]$. By (2.11), for $i = 1, 2$,

$$\begin{aligned} M_{y_i y_i} &= \frac{p\omega^{p-2} R_1}{y_3} \left[(\omega R_2 + (p-1)R_1)(H')^2 + \omega y_3 H''\right] \\ &= \frac{p\omega^{p-2}}{y_3(\Phi')^3} \left[\Lambda(H')^2 + \omega y_3 H''(\Phi')^2\right], \end{aligned}$$

giving sign $M_{y_i y_i} = (-1)[-\operatorname{sign}(p-2)]$, since $\Phi' < 0$. □

Now that we have a partial Bellman function candidate for $1 < p < 2$, from Case (3_2), satisfying all of the properties of the Bellman function, including restrictive concavity, we can turn our attention to Case (2_2). From Case (2_2) we will get a Bellman candidate on all Ξ_+, or part of it, depending on the τ- and p-values. The partial Bellman candidate, from Case (2_2), turns out to be the missing half for Case (3_2). We already have the solution for Case (2) from Lemma 21, but the value of the constant is needed before we can progress further.

2.5.2 Case (2) for $1 < p < 2$

Lemma 53. *If $1 < p < 2$, then in Case (2_2), the value of the constant in Lemma 21 is* $c = \left(\frac{1}{(p-1)^2} + \tau^2\right)^{p/2} = \left((p^* - 1)^2 + \tau^2\right)^{p/2}$.

Proof. If $M(y) = (1+\tau^2)^{\frac{p}{2}}[y_1^2 + 2\gamma y_1 y_2 + y_2^2]^{\frac{p}{2}} + c[y_3 - (y_1 - y_2)^p]$ (where $\gamma = \dfrac{1-\tau^2}{1+\tau^2}$) is to be a candidate or partial candidate, then it must agree, at $y_2 = \frac{2-p}{p} y_1$, with the solution M given implicitly by the relation $G(y_1 - y_2, y_1 + y_2) = y_3 G(1, \sqrt{\omega^2 - \tau^2})$, from Proposition 46. At $y_2 = \frac{2-p}{p} y_1$,

$$
\begin{aligned}
(\sqrt{\omega^2 - \tau^2} + 1)^{p-1}[1 - (p-1)\sqrt{\omega^2 - \tau^2}] &= G(1, \sqrt{\omega^2 - \tau^2}) \\
&= \frac{1}{y_3}(2y_1)^{p-1}[(2-p)y_1 + (p-2)y_1] = 0.
\end{aligned}
$$

Since $\sqrt{\omega^2 - \tau^2} + 1 \neq 0$, $\sqrt{\omega^2 - \tau^2} = \frac{1}{p-1}$, which implies $\omega = \left(\frac{1}{(p-1)^2} + \tau^2\right)^{\frac{1}{2}}$. So,

$$
\begin{aligned}
\left(\frac{1}{(p-1)^2} + \tau^2\right)^{\frac{p}{2}} y_3 &= \omega^p y_3 \\
&= M(y_1, \frac{2-p}{p} y_1, y_3) \\
&= \left[\left(\frac{2}{p} y_1\right)^2 + \tau^2 \left(\frac{2(p-1)}{p} y_1\right)^2\right]^{\frac{2}{p}} + c\left[y_3 - \left(\frac{2(p-1)}{p} y_1\right)^p\right].
\end{aligned}
$$

Now just solve for c. $\square$

In the following Lemma the value of τ has to be restricted to $\tau^2 \leq \frac{1}{2p-1}$, so that restrictive concavity is satisfied for our Bellman candidate. Actually, the τ-values play an even bigger role. Depending on the value of $(\tau, p) \in [-1, 1] \times (1, 2)$, there is either one or two Bellman function candidates. For $(\tau, p) \in B$, from Figure 2.10, there is a partial Bellman candidate arising from Case (2_2). So we can glue this together with the other partial candidate obtained in Case (3_2). This gives a Bellman candidate, as before, having characteristics as in Figure 2.6. For $(\tau, p) \in A \cup C$ the candidate obtained from Case (2_2) maintains restrictive concavity throughout Ξ_+ and is therefore requires no gluing. To avoid the difficulty of determining which candidate to choose and how to determine the optimal constant from Case (2), we restrict (τ, p) to region B.

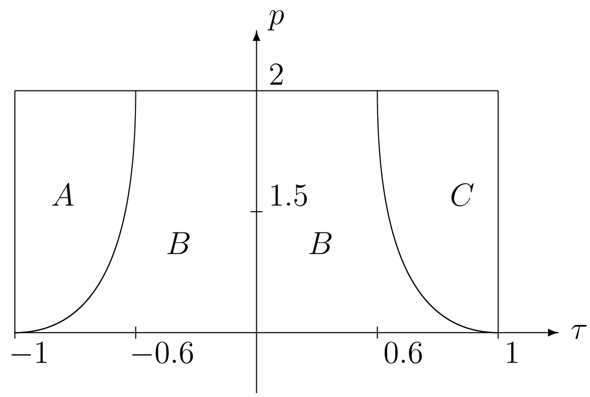

Figure 2.10: Splitting $[-1,1]\times(1,2)$ in the $(\tau\times p)$–plane into three regions A, B and C. Region $B = \left\{\tau^2 \le \frac{1}{2p-1}\right\}$.

Recall that the partial Bellman candidate, M, obtained from Case (2_2), for $1 < p < 2$, satisfies $M_{y_i y_3} = M_{y_3 y_3} = 0$ and hence $D_i = 0$, for $i = 1, 2$. So all that still needs to be checked for restrictive concavity is the sign of $M_{y_1 y_1}$ and $M_{y_2 y_2}$. Since $M_{y_1 y_1} \le M_{y_2 y_2}$, we just need to show that $M_{y_2 y_2} \le 0$ on $\frac{2-p}{p} y_1 \le y_2 \le y_1$ in Ξ_+. This is considered in the following Lemmas.

Lemma 54. *In Case* (2_2), $M_{y_2 y_2}(y_1, \frac{2-p}{p} y_1, y_3) \le 0$ *for* $\tau^2 \le \frac{1}{2p-1}$ *and* $1 < p < 2$.

Proof. The solution M that we get from (2_2), when $1 < p < 2$, is obtained from Lemmas 21 and 53. Let $\gamma = \dfrac{1-\tau^2}{1+\tau^2}$, $f_1(y) = y_1^2 + y_2^2 + 2\gamma y_1 y_2$, $f_2(y) = (p-2)(y_2 + \gamma y_1)^2 + f_1(y)$ and $f_3(y) = y_1 - y_2$. Then

$$M_{y_2 y_2} = p(1+\tau^2)^{\frac{p}{2}} f_1^{\frac{p-4}{2}} f_2 - p(p-1)\left(\frac{1}{(p-1)^2} + \tau^2\right)^{\frac{p}{2}} f_3^{p-2}.$$

To show that $M_{y_2 y_2}(y_1, \frac{2-p}{p} y_1, y_3) \le 0$ we change to x-variables, make the substitution $s = \dfrac{x_2}{x_1}$ and multiply both sides of the inequality by $(s^2+\tau^2)^{\frac{4-p}{2}}$. Denote $F(s)$ as the left side of the new inequality. So we just need to show that

$$F(s) = (p-2)(s-\tau^2)^2 + (1+\tau^2)(s^2+\tau^2) - (p-1)((p^*-1)^2+\tau^2)^{\frac{p}{2}}(s^2+\tau^2)^{\frac{4-p}{2}} \leq 0,$$

at $s = p^* - 1$. Or equivalently we need to show that

$$\tau^2((p^*-1)^2+\tau^2) \leq (p^*-1-\tau^2)^2. \tag{2.25}$$

Note that (2.25) reduces to $\tau^2 \leq \frac{1}{2p-1}$. Therefore, $M_{y_2y_2}(y_1, \frac{2-p}{p}y_1, y_3) \leq 0$ for $\tau^2 \leq \frac{1}{2p-1}$ and $1 < p < 2$. $\square$

Lemma 55. *In Case* $(2_2), M_{y_2y_2}(y_1, cy_1, y_3) \leq 0$, *for all* $c \in \left[\frac{2-p}{p}, 1\right]$, *for* $\tau^2 \leq \frac{1}{2p-1}$ *and* $1 < p < 2$.

Proof. Using $M_{y_2y_2}$ from Lemma 54 we see that $M_{y_2y_2} \leq 0$ is equivalent to

$$(1+\tau^2)^{\frac{p}{2}} f_3^{2-p} f_2 f_1^{\frac{p-4}{2}} - p(p-1)\left(\frac{1}{(p-1)^2}+\tau^2\right)^{\frac{p}{2}} \leq 0.$$

Observe that the function $f_2/(f_1^{\frac{4-p}{2}})$ is strictly positive, has a horizontal asymptote at the y_2-axis, increases on $(-\infty, -\gamma)$, and decreases on $(-\gamma, \infty)$. As y_2 increases from $\frac{2-p}{p}$ to 1, f_3^{2-p} and $f_2 f_1^{\frac{p-4}{2}}$ both decrease. Since $M_{y_2y_2}(y_1, \frac{2-p}{p}y_1, y_3) \leq 0$ (as shown in Lemma 54), the result follows here as well. $\square$

Lemma 56. *The Monge-Ampère solution in Case* (2_2) *yields the following results for* $1 < p < 2$. $M_{y_2y_2}(y_1, y_1, y_3) < 0$ *for* $|\tau| \leq 1$ *and* $M_{y_2y_2}(y_1, -y_1, y_3) > 0$ *for* $|\tau| \leq \frac{1}{2}$

Proof. Let f_1, f_2 and f_3 be as in Lemma 54 and

$$g = (1+\tau^2)^{\frac{p}{2}} f_3^{2-p} f_2 - (p-1)\Big(\frac{1}{(p-1)^2} + \tau^2\Big)^{\frac{p}{2}} f_1^{\frac{4-p}{2}}.$$

Note that $M_{y_2 y_2}$ and g have the same signs. It is clear that $g(y_1, y_1, y_3) < 0$, proving the first inequality. One can now verify that $g(y_1, -y_1, y_3) > 0$ for $|\tau| \leq \frac{1}{2}$ which proves the second inequality. $\square$

Remark 57. One can see in the graph of $y_1^{2-p} g(y_1, y_1, y_3)$ that $g(y_1, y_1, y_3) < 0$, in regions A and C, (see Figure 2.10). This tells us that the Bellman candidate from Case (2_2) will maintain restrictive concavity throughout the domain in for $(\tau, p) \in A \cup C$. Furthermore, there will be an improvement in the constant $\big((p^* - 1)^2 + \tau^2\big)^{\frac{p}{2}}$ that can still be used to still maintain restrictive concavity in $A \cup C$.

By Lemmas 55 and 56 we obtain a partial Bellman candidate from Case (2_2), when $1 < p < 2$ and τ satisfying $\tau^2 \leq \frac{1}{2p-1}$. As before, we will glue this partial candidate from Case (2_2) to the partial candidate in Case (3_2) to obtain the Bellman candidate for $1 < p < 2$. This glued function will be a Bellman candidate since it satisfies all Bellman function properties, including the estimates needed for restrictive concavity. However, we must choose the smaller set of τ restrictions between the implicit and explicit part of the candidate, so that restrictive concavity will be satisfied. In particular we must restrict τ such that $\tau^2 \leq \frac{1}{2p-1}$.

2.6 Remaining cases and why they do not give the Bellman function candidate

Now that we have particular cases in which the Monge-Ampère solution gives a Bellman function candidate, we would like to discuss the remaining cases. It can be shown that all remaining cases do not yield a Bellman function candidate, except for Case (4) which is still not determined.

2.6.1 Case (1_2) for $1 < p < 2$ and Case (3_2) for $2 < p < \infty$

It was shown in Lemmas 20, 52 that the Monge-Ampère solution obtained in each case does not have the appropriate restrictive concavity property to be a Bellman function candidate. We mention this here again simply for clarity.

2.6.2 Case (1_1)

We can consider Cases (1_1) and (3_1) simultaneously, for part of the calculation, since the same argument will work in both cases. In both cases, y_2 is fixed and the Monge-Ampère solution is given by $M(y) = t_1 y_1 + t_3 y_3 + t_0$ on the characteristics $dt_1 y_1 + dt_3 y_3 + dt_0 = 0$. As shown in Figure 2.11, $y_2 \geq 0$ in case (1_2) and $y_2 \leq 0$ in Case (3_2), since if not then the characteristics go outside of the domain Ξ_+.

Lemma 58. *In Cases* (1_1) *and* (3_1), *the solution to the Monge-Ampère can be written as,*

$$M(y) = \left(\frac{\sqrt{(u + y_2)^2 + \tau^2 (u - y_2)^2}}{u - y_2} \right)^p y_3$$

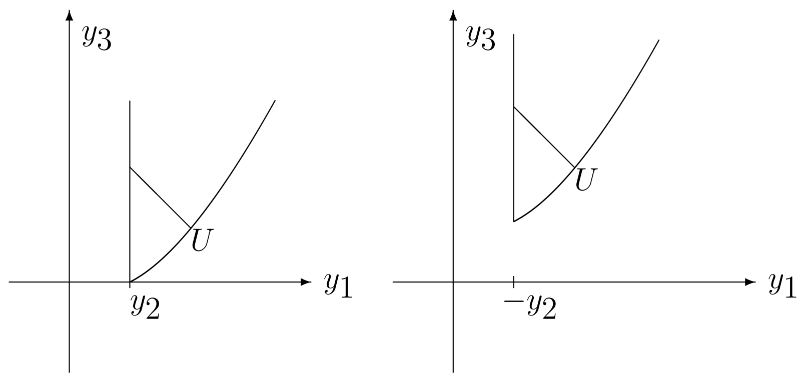

Figure 2.11: Sample characteristic for Monge-Ampère solution in Cases (1_1) and (3_1)

where $u = u(y_1, y_2, y_3)$ *is the solution to the equation* $\frac{y_1+\left(\frac{2}{p}-1\right)|y_2|}{y_3} = \frac{u+\left(\frac{2}{p}-1\right)|y_2|}{(u-y_2)^p}$.

Proof. Any characteristic, in Cases (1_1) and (3_1), go from $U = (u, y_2, (u-y_2)^p)$ to $W = (|y_2|, y_2, w)$. Throughout the proof we will use the properties of the Bellman function derived in Proposition 7. Using the Neumann property and the property from Proposition 9 we get $y_2 M_{y_2} = y_1 M_{y_1} = |y_2| t_1$ at W. By homogeneity at W we get

$$p|y_2|t_1 + pwt_3 + pt_0 = pM(W) = y_1 M_{y_1} + y_2 M_{y_2} + py_3 M_{y_3} = 2t_1|y_2| + pwt_3.$$

Following the same argument as in Lemma 13, gives

$$M(y) = \left(\frac{\sqrt{(u+y_2)^2 + \tau^2(u-y_2)^2}}{u-y_2}\right)^p y_3,$$

where $u = u(y_1, y_2, y_3)$ is the solution to the equation

$$\frac{y_1 + \left(\frac{2}{p}-1\right)|y_2|}{y_3} = \frac{u + \left(\frac{2}{p}-1\right)|y_2|}{(u-y_2)^p}. \tag{2.26}$$

Since the solution, M, does not satisfy the restrictive concavity property necessary to be the Bellman function (as we will soon show), we are not concerned about existence of the solution u in equation (2.26). $\square$

Lemma 59. *If* $\omega = \left(\frac{M(y)}{y_3}\right)^{\frac{1}{p}}$*, then in Cases* (1_1) *and* (3_1)*, the solution* u *to equation (2.26) can be expressed as* $u = \frac{\sqrt{\omega^2-\tau^2}+1}{\sqrt{\omega^2-\tau^2}-1}y_2$ *and equation (2.26) can be rewritten as*

$$2^p|y_2|^{p-1}[py_1 + (2-p)|y_2|] = y_3|\beta - 1|^{p-1}[p(\beta+1) + (2-p)|\beta - 1|], \tag{2.27}$$

where $\beta = \sqrt{\omega^2 - \tau^2}$*. Furthermore,* $\operatorname{sign} y_2 = \operatorname{sign}(\beta - 1)$.

Proof. Let us show that $u = \frac{\sqrt{\omega^2-\tau^2}+1}{\sqrt{\omega^2-\tau^2}-1}y_2$ first. This follows from inverting

$$om = \frac{\sqrt{(u+y_2)^2 + \tau^2(u-y_2)^2}}{u - y_2},$$

and using the properties $\omega \geq |\tau|$ and $u \pm y_2 \geq 0$. Now that we have $u = \frac{\sqrt{\omega^2-\tau^2}+1}{\sqrt{\omega^2-\tau^2}-1}y_2$, we can use it to get the next result. Note that $u \geq 0$ and $\sqrt{\omega^2 - \tau^2} \geq 0$, which implies that $\operatorname{sign} y_2 = \operatorname{sign}(\sqrt{\omega^2 - \tau^2} - 1)$. To get (2.27), simply plug $u = \frac{\sqrt{\omega^2-\tau^2}+1}{\sqrt{\omega^2-\tau^2}-1}y_2$ in equation (2.26). $\square$

We can no longer discuss Cases (1_1) and (3_1) together, so for the remainder of the Subsection the focus will be on Case (1_1) only.

Lemma 60. *In Case* (1_1)*, the solution* M *from Lemma 58 can be rewritten in the implicit form*

$$G(y_2 + y_1, y_2 - y_1) = y_3 G(\sqrt{\omega^2 - \tau^2}, -1),$$

where $G(z_1, z_2) = (z_1 + z_2)^{p-1}[z_1 - (p-1)z_2]$.

Proof. Recall that for Case (1_1) we have $y_2 > 0$.

$$y_2 = \frac{1}{2}(x_2 - x_1) > 0 \quad \Longrightarrow \quad x_2 > x_1$$
$$\operatorname{sign}(\sqrt{\omega^2 - \tau^2} - 1) = \operatorname{sign} y_2 > 0 \quad \Longrightarrow \quad \sqrt{\omega^2 - \tau^2} > 1 \quad \Longrightarrow \quad \omega > \sqrt{\tau^2 + 1}$$

So, $\mathcal{B}(x) = M(y) > y_3(\tau^2 + 1)^{\frac{p}{2}}$. Now (2.27) can be rewritten as

$$(x_2 - x_1)^{p-1}[(p-1)x_1 + x_2] =$$
$$\left[\sqrt{\mathcal{B}^{2/p} - \tau^2 x_3^{2/p}} - x_3^{1/p}\right]^{p-1}\left[\sqrt{\mathcal{B}^{2/p} - \tau^2 x_3^{2/p}} + (p-1)x_3^{1/p}\right].$$

Therefore,

$$G(x_2, -x_1) = G\Big(\sqrt{\mathcal{B}^{2/p} - \tau^2 x_3^{2/p}}, -x_3^{2/p}\Big)$$

or by factoring out $x_3^{\frac{1}{p}}$ on the right side we get

$$G(y_2 + y_1, y_2 - y_1) = y_3 G(\sqrt{\omega^2 - \tau^2}, -1). \qquad \square$$

Recall that the Monge-Ampère solution must satisfy the restrictive concavity conditions in Proposition 6 to be a Bellman function candidate. We will show that the Monge-Ampère solution obtained in Case (1_1) has $D_1 < 0$ and therefore cannot be a Bellman candidate.

Lemma 61. *In Case* (1_1) *we choose* $H(y_1, y_2) = G(y_1 + y_2, -y_1 + y_2)$ *because of how the implicit solution is defined and obtain* $\operatorname{sign} H'' = \operatorname{sign}(p-2)$

Proof. We already computed

$$H'' = \begin{cases} 4G_{z_1 z_2}, & \alpha_j = \beta_j \\ 0, & \alpha_j = -\beta_j \end{cases}$$

in Lemma 16.

Since, $\alpha_1 = 1$ since $\alpha_2 = 1$ since $\beta_1 = -1$ and since $\beta_2 = 1$,

$$G_{z_1 z_2} = p(p-1)(p-2)(y_1 - y_2)(2y_2)^{p-3}. \qquad \square$$

Lemma 62. *If $p \neq 2$ then $D_2 < 0$ in Case (1_1) for all τ.*

Proof. We use the partial derivatives of G from the proof of Lemma 16 to make the computations of Φ' and Φ'' easier. Let $\alpha_p = \frac{p(p-1)\omega(\beta-1)^{p-3}}{\beta^3}$ and $\beta = \sqrt{\omega^2 - \tau^2}$.

$$\begin{aligned}
\Phi(\omega) &= G(\beta, 1) \\
\Phi'(\omega) &= p[\beta - 1]^{p-2}[\omega + (p-2)\omega\beta^{-1}] \\
\Phi''(\omega) &= G_{z_1 z_2}(\beta, -1)\beta^{-2}\omega^2 + G_{z_1}(\beta, -1)[-\omega\beta^{-3} + \beta^{-1}] \\
&= p(p-1)[\beta - 1]^{p-3}[\beta + p - 3]\frac{\omega^2}{\beta^2} - p\frac{\tau^2}{\beta^3}[\beta - 1]^{p-2}[\beta + p - 2] \\
\Lambda &= (p-1)\Phi' - \omega\Phi'' \\
&= \alpha_p[(\beta - 1)\beta^3(\beta^{-1}(p-2) + 1) - \omega^2\beta(\beta + p - 3) + \tau^2(\beta - 1)(\beta + p - 2)] \\
&= \alpha_p[(\beta^2 + \tau^2)(\beta - 1)(\beta + p - 2) - \omega^2\beta(\beta + p - 3)] \\
&= \alpha_p\omega^2[\beta^2 + \beta(p-2) - \beta - (p-2) - \beta^2 - \beta(p-3)] \\
&= -\frac{p(p-1)(p-2)\omega^3(\sqrt{\omega^2 - \tau^2} - 1)^{p-3}}{(\sqrt{\omega^2 - \tau^2})^3}
\end{aligned}$$

From Lemma 59, $\operatorname{sign}(\beta - 1) = \operatorname{sign} y_2 > 0$ and $\omega^2 > \tau^2 > 0$. Therefore, by Lemma 61 and (2.12) $\operatorname{sign} D_2 = \operatorname{sign} H'' \operatorname{sign} \Lambda = -(\operatorname{sign}(p-2))^2 < 0$. □

Since $D_2 < 0$ in Case (1_1), we get the following result.

Proposition 63. *Case* (1_1) *does not give a Bellman function candidate.*

2.6.3 Case (3_1) does not provide a Bellman function candidate

Much of the work needed to show that the Monge-Ampère solution cannot be the Bellman function, in Case (3_1), has already been started in Section 2.6.2. Let us finish the argument.

Lemma 64. *In Case* (3_1)*, the solution* M *from Lemma 58 can be rewritten in the implicit form* $G(y_2 - y_1, -y_1 - y_2) = y_3 G(1, -\sqrt{\omega^2 - \tau^2})$*, where* $G(z_1, z_2) = (z_1 + z_2)^{p-1}[z_1 - (p-1)z_2]$.

Proof. Recall that in Case (3_2) we have that $y_2 < 0$.

$$y_2 = \frac{1}{2}(x_2 - x_1) < 0 \quad \Longrightarrow \quad x_2 < x_1$$
$$\operatorname{sign}(\sqrt{\omega^2 - \tau^2} - 1) = \operatorname{sign} y_2 < 0 \quad \Longrightarrow \quad \sqrt{\omega^2 - \tau^2} < 1 \quad \Longrightarrow \quad \omega < \sqrt{\tau^2 + 1}.$$

So, $\mathcal{B}(x) = M(y) < y_3(\tau^2 + 1)^{\frac{p}{2}}$. Now (2.27) can be rewritten as

$$(x_1 - x_2)^{p-1}[x_1 + (p-1)x_2] = \Big[x_3^{1/p} - \sqrt{\mathcal{B}^{2/p} - \tau^2 x_3^{2/p}}\Big]\Big[(p-1)\sqrt{\mathcal{B}^{2/p} - \tau^2 x_3^{2/p}} + x_3^{1/p}\Big].$$

Therefore,

$$G(x_1, -x_2) = G\Big(x_3^{1/p}, -\sqrt{\mathcal{B}^{2/p} - \tau^2 x_3^{2/p}}\Big),$$

or by factoring out $x_3^{1/p}$ on the right side we get

$$G(y_1 - y_2, -y_1 - y_2) = y_3 G(1, -\sqrt{\omega^2 - \tau^2}). \qquad \square$$

Since y_2 is fixed, $D_2 \geq 0$ must be true in order that the Monge-Ampère solution from Case (3_1) is the Bellman function (see Proposition 6). However, the contrary is true: $D_2 < 0$.

Lemma 65. *In Case* (3_1) *we choose* $H(y_1, y_2) = G(y_1 - y_2, -y_1 + y_2)$ *because of how the implicit solution is defined and obtain* $\operatorname{sign} H'' = \operatorname{sign}(p-2)$

Proof. We already computed

$$H'' = \begin{cases} 4G_{z_1 z_2}, & \alpha_j = \beta_j \\ 0, & \alpha_j = -\beta_j \end{cases}$$

in Lemma 16.

Since, $\alpha_1 = 1, \alpha_2 = -1, \beta_1 = -1$ and $\beta_2 = 1$, it follows that

$$G_{z_1 z_2} = p(p-1)(p-2)(y_1 - y_2)(2y_2)^{p-3}. \qquad \square$$

Lemma 66. *If* $p \neq 2$*, then* $D_2 < 0$ *in Case* (3_1) *for all* τ.

Proof. We use the partial derivatives of G computed in the proof of Lemma 16 to make the following computations of Φ' and Φ'' easier. Let $\beta = \sqrt{\omega^2 - \tau^2}$.

$$\Phi(\omega) = G(1, -\beta)$$

$$\Phi'(\omega) = -p(p-1)\omega(1-\beta)^{p-2}$$

$$\Phi''(\omega) = -p(p-1)[(1-\beta)^{p-2} - (p-2)\omega^2\beta^{-1}]$$

$$\begin{aligned}
\Lambda &= (p-1)\Phi' - \omega\Phi'' \\
&= p(p-1)\omega(1-\beta)^{p-3}[-(p-1)(1-\beta) + (1-\beta) - (p-2)\omega^2\beta^{-1}] \\
&= -p(p-1)\omega(1-\beta)^{p-3}(p-2)[1-\beta+\omega^2\beta^{-1}] \\
&= -p(p-1)(p-2)\omega(1-\beta)^{p-3}\left(1+\frac{\tau^2}{\beta}\right).
\end{aligned}$$

From Lemma 59, $1-\beta > 0$ and $\omega^2 > \tau^2 > 0$. Therefore, by Lemma 65 and (2.12) $\operatorname{sign} D_2 = \operatorname{sign} H'' \operatorname{sign}\Lambda = -(\operatorname{sign}(p-2))^2 < 0$. $\square$

Having shown that $D_2 < 0$ in Case (3_1) implies that the Monge-Ampère solution in that case cannot be the Bellman function.

Proposition 67. *Case* (3_1) *does not give a Bellman function candidate.*

2.6.4 Case(2_1) gives a partial Bellman function candidate

Case (2) was considered without having to fix either y_1 or y_2 first, so there is nothing new to do here. Refer to Sections 2.5.2 and 2.2.1.3 for more details.

2.6.5 Case (4) may or may not yield a Bellman function candidate

For $\tau = 0$, it was shown in [36] that Case (4) does not produce a Bellman function candidate, since some simple extremal functions give a contradiction to linearity of the Monge-Ampère solution on characteristics. However, for $\tau \neq 0$ it is much more difficult to show this. Those same extremal functions do contradict linearity for some p-values and some signs of the Martingale transform. For the sign of the Martingale transform where we do not have a

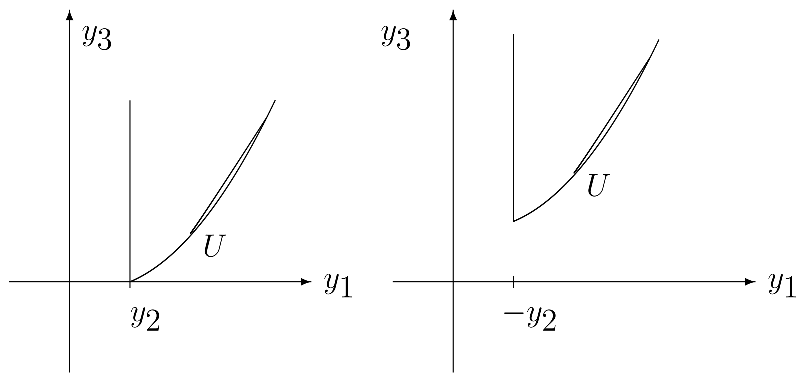

Figure 2.12: Characteristic of solution in Case (4_1).

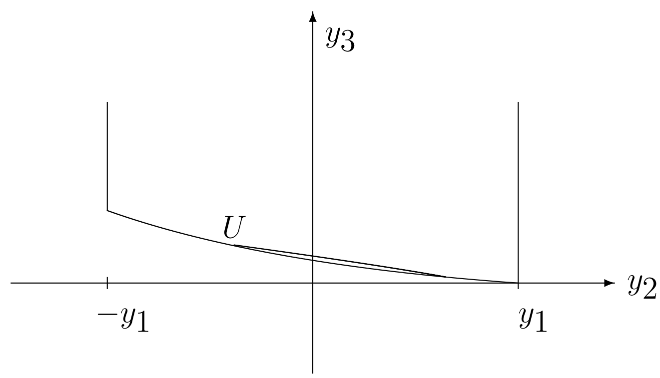

Figure 2.13: Characteristic for the solution from Case (4_2)

contradiction, a new set of test of extremal functions would have to be found. Since the Bellman function has already been constructed from other cases, this case has not been investigated any further than just described. So, for p and τ values not mentioned in the main result, Case (4) could give a Bellman candidate throughout Ξ_+ or we could get a partial Bellman candidate that may work well with the characteristics from Case (2_1).

Chapter 3

Laminates Meet Burkholder Functions

3.1 Introduction

In Chapter 2 the L^p-operator norm was computed for a "quadratic perturbation" of the martingale transform using the Bellman function technique, which is similar to how Burkholder originally did, for $\tau = 0$, in [15]. By "quadratic perturbation", we are referring to the quantity $(Y^2 + \tau^2 X^2)^{\frac{1}{2}}$, where $\tau \in \mathbb{R}$ is small, X is a martingale and Y is the corresponding martingale transform.

The method of Bourgain [11], for the Hilbert transform, which was later generalized for a large class of Fourier multiplier operators by Geiss, Montgomery-Smith, Saksman [24], is to discretize the operator and generalize it to a higher dimensional setting. This operator in the higher dimensional setting will turn out to have the same operator norm and it naturally connects with discrete martingales, if done in a careful and clever way. In the end, one has the operator norm of the singular integral bounded below by the operator norm of the martingale transform, which Burkholder found in [15]. This approach can be used for estimating $\left\|\left(R_1^2 - R_2^2, \tau I\right)\right\|_{L^p \to L^p}$ from below as well, see [10]. However, we will present an entirely different approach to the problem.

Rather than working with estimates on the martingale transform, we only need to consider the "Burkholder" functions that were used to find those sharp estimates on the martingale transform. More specifically, we analyze the behavior of the "Burkholder" functions, U and v found in Chapter 2 (and reiterated in Definition 76 and Theorem 78), associated with determining the L^p-operator norm of the quadratic perturbation of the martingale transform. Using the fact that U is the least bi-concave majorant of v (in the appropriately chosen coordinates), in addition to some of the ways in which the two functions interact will allow us to construct an appropriate sequence of laminates, which approximate the push forward of the 2-dimensional Lebesgue measure by the Hessian of a smooth function with compact support. Once the appropriate sequence of laminates is constructed, we are finished since Riesz transforms can be written as fractional derivatives of some smooth function. The beauty of this method is that it quickly gets us the sharp lower bound constant with very easy calculations. This lower bound argument is discussed in Section 3.2.

The Burkholder functions U and v also play a crucial role in obtaining the sharp upper bound estimate as well. With the Burkholder functions we are able to extend sharp estimates of $(Y^2 + \tau^2 X^2)^{\frac{1}{2}}$, obtained in Chapter 2, from the discrete martingale setting to the continuous martingale setting. The use of "heat martingales", as in [2] and [3], will allow us to connect the Riesz transforms to the continuous martingales estimate, without picking up any additional constants. This upper bound argument is presented in Section 3.4.

3.2 Lower Bound Estimate

3.2.1 Laminates and gradients

We denote by $\mathbb{R}^{m\times n}$ the space of real $m \times n$ matrices and by $\mathbb{R}^{n\times n}_{sym}$ the set of real $n \times n$ symmetric matrices.

Definition 68. We say that a function $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ is *rank-one convex*, if $t \mapsto f(A+tB)$ is convex for all $A, B \in \mathbb{R}^{m\times n}$ with rank $B = 1$.

Let $\mathcal{P}(\mathbb{R}^{m\times n})$ denote the set of all compactly supported probability measures on $\mathbb{R}^{m\times n}$. For $\nu \in \mathcal{P}$ we denote by $\overline{\nu} = \int_{\mathbb{R}^{m\times n}} X \, d\nu(X)$ the center of mass or *barycenter* of ν.

Definition 69. A measure $\nu \in \mathcal{P}$ is called a *laminate*, denoted $\nu \in \mathcal{L}$, if

$$f(\overline{\nu}) \leq \int_{\mathbb{R}^{m\times n}} f \, d\nu \tag{3.1}$$

for all rank-one convex functions f. The set of laminates with barycenter 0 is denoted by $\mathcal{L}_0(\mathbb{R}^{m\times n})$.

Laminates play an important role in several landmark applications of convex integration for producing unexpected counterexamples, see for instance [30, 27, 1, 34, 21]. For our purposes the case of 2×2 symmetric matrices is of relevance, therefore in the following we restrict attention to this case. The key point is that laminates can be viewed as probability measures recording the gradient distribution of maps, see Theorem 72 below. This is by now a very standard technique. Refer to [9] for the main steps of the argument. Detailed proofs of these statements can be found for example in [30, 26, 34].

Definition 70. Given a set $U \subset \mathbb{R}^{2\times 2}$ we call $\mathcal{PL}(U)$ the set of *prelaminates* generated in U. This is the smallest class of probability measures on $\mathbb{R}^{2\times 2}$ which

- contains all measures of the form $\lambda\delta_A + (1-\lambda)\delta_B$ with $\lambda \in [0,1]$ and $\operatorname{rank}(A-B) = 1$;
- is closed under splitting in the following sense if $\lambda\delta_A + (1-\lambda)\tilde{\nu}$ belongs to $\mathcal{PL}(U)$ for some $\tilde{\nu} \in \mathcal{P}(\mathbb{R}^{2\times 2})$ and μ also belongs to $\mathcal{PL}(U)$ with $\overline{\mu} = A$, then also $\lambda\mu + (1-\lambda)\tilde{\nu}$ belongs to $\mathcal{PL}(U)$.

The order of a prelaminate denotes the number of splittings required to obtain the measure from a Dirac measure.

Example 71. The measure

$$\frac{1}{4}\delta_{\mathrm{diag}(1,1)} + \frac{1}{4}\delta_{\mathrm{diag}(-1,1)} + \frac{1}{2}\delta_{\mathrm{diag}(0,-1)},$$

where

$$\mathrm{diag}(x,y) := \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix},$$

is a second order prelaminate with barycenter 0.

It is clear from the definition that $\mathcal{PL}(U)$ consists of atomic measures. Also, from a repeated application of Jensen's inequality it follows that $\mathcal{PL} \subset \mathcal{L}$. The following statement, links laminates supported on symmetric matrices with second derivatives of functions.

Theorem 72. *Let $\nu \in \mathcal{L}_0(\mathbb{R}^{2\times 2}_{sym})$. Then there exists a sequence $u_j \in C_c^\infty(B_1(0))$ with uniformly bounded second derivatives, such that*

$$\int_{B_1(0)} \phi(D^2 u_j(x))\,dx \to \int_{\mathbb{R}^{2\times 2}_{sym}} \phi\,d\nu$$

for all continuous $\phi : \mathbb{R}^{2\times 2}_{sym} \to \mathbb{R}$.

3.2.2 Laminates and lower bounds

Let $\tau \in \mathbb{R}$ be fixed. Our goal is to find

$$\sup_{\varphi \in \mathcal{S}(\mathbb{R}^2)} \frac{\left\| \left((R_1^2\varphi - R_2^2\varphi)^2 + \tau^2 (R_1^2\varphi + R_2^2\varphi)^2 \right)^{1/2} \right\|_p}{\|R_1^2\varphi + R_2^2\varphi\|_p}, \tag{3.2}$$

for the planar Riesz transforms R_1 and R_2, where $\mathcal{S}(\mathbb{R}^2)$ is the Schwartz class. We can rework the Riesz transforms acting on φ into the second derivative of a function $u \in \mathcal{S}(\mathbb{R}^2)$ in the following way.

$$R_i^2\varphi = \left(-\frac{\xi_i^2}{|\xi|^2}\widehat{\varphi} \right)^{\vee} = \partial_i^2 u,$$

where "$\widehat{\ }$" denotes the Fourier transform, "$\vee$" denotes the inverse Fourier transform and $-\Delta u = \varphi$. So (3.2) is equivalent to

$$\sup_{u \in \mathcal{S}(\mathbb{R}^2)} \frac{\int |(\partial_{11}^2 u - \partial_{22}^2 u)^2 + \tau^2 (\partial_{11}^2 u + \partial_{22}^2 u)^2|^{\frac{p}{2}}}{\int |\partial_{11}^2 u + \partial_{22}^2 u|^p}. \tag{3.3}$$

Let A_{ij} denote, as usual, the ij-entry of a matrix A and put

$$\begin{aligned} \phi_1(A) &= |(A_{11} - A_{22})^2 + \tau^2 (A_{11} + A_{22})^2|^{\frac{p}{2}}, \\ \phi_2(A) &= |A_{11} + A_{22}|^p. \end{aligned} \tag{3.4}$$

Using a standard cut-off argument we can write replace $\mathcal{S}(\mathbb{R}^2)$ with $C_c^\infty(\mathbb{R}^2)$ and write (3.3) as

$$\sup_{u\in C_c^\infty(\mathbb{R}^2)} \frac{\int \phi_1(D^2u)\,dx}{\int \phi_2(D^2u)\,dx}. \tag{3.5}$$

From Corollary 72 we deduce that

$$\sup_{u\in C_c^\infty(\mathbb{R}^2)} \frac{\int \phi_1(D^2u)\,dx}{\int \phi_2(D^2u)\,dx} \quad \geq \sup_{\nu\in\mathcal{L}_0(\mathbb{R}^{2\times 2}_{sym})} \frac{\int \phi_1\,d\nu}{\int \phi_2\,d\nu}. \tag{3.6}$$

Our goal is therefore to prove the following

Theorem 73. *For any $1 < p < \infty$ and $\tau \in \mathbb{R}$ there exists a sequence $\nu_N \in \mathcal{L}_0(\mathbb{R}^{2\times 2}_{sym})$ such that*

$$\frac{\int \phi_1\,d\nu_N}{\int \phi_2\,d\nu_N} \to ((p^*-1)^2+\tau^2)^{\frac{p}{2}}.$$

3.2.3 Proof of Theorem 73

A function $f(x,y)$ of two variables is said to be biconvex if the functions $x \mapsto f(x,y)$ and $y \mapsto f(x,y)$ are convex for all x,y. We start with the following inequality for biconvex functions in the plane.

Lemma 74. *Let $k \in (-1,1)$ and $N > 1$. For every $f \in C(\mathbb{R}^2)$ biconvex we have*

$$f(1,1) \leq \frac{1}{1-k}\int_1^N \big(f(kt,t)+f(t,kt)\big)t^{-\frac{2}{1-k}}\,\frac{dt}{t} + f(N,N)N^{-\frac{2}{1-k}}. \tag{3.7}$$

Proof. By a standard regularization argument it suffices to show the inequality for $f \in$

$C^1(\mathbb{R}^2)$ biconvex. The biconvexity implies the following elementary inequalities:

$$f(t,t) \leq \lambda^\epsilon f(t, t+\epsilon) + (1-\lambda^\epsilon) f(t, kt), \tag{3.8}$$

$$f(t, t+\epsilon) \leq \mu^\epsilon f(t+\epsilon, t+\epsilon) + (1-\mu^\epsilon) f(k(t+\epsilon), t+\epsilon), \tag{3.9}$$

where

$$\lambda^\epsilon = 1 - \frac{\epsilon}{t(1-k)+\epsilon}$$
$$\mu^\epsilon = 1 - \frac{\epsilon}{t(1-k)+\epsilon(1-k)}.$$

Combining (3.8) and (3.9) and observing that $\lambda^\epsilon, \mu^\epsilon = 1 - \frac{\epsilon}{t(1-k)} + o(\epsilon)$, we obtain

$$\begin{aligned} \frac{f\big(t+\epsilon, t+\epsilon\big) - f(t,t)}{\epsilon} &- \frac{2}{t(1-k)} f(t+\epsilon, t+\epsilon) \geq \\ &- \frac{1}{t(1-k)} \Big(f(k(t+\epsilon), t+\epsilon) + f(t, kt) \Big) + o(1). \end{aligned} \tag{3.10}$$

Letting $\epsilon \to 0+$ this yields

$$-\frac{\partial}{\partial t} f(t,t) + \frac{2}{t(1-k)} f(t,t) \leq \frac{1}{t(1-k)} \big(f(kt, t) + f(t, kt) \big)$$

Multiplying both sides by $t^{-\frac{2}{1-k}}$ and integrating, we obtain (3.7) as required. $\square$

The method of obtaining continuous laminates by integrating a differential inequality as above is due to Kirchheim, and appeared first in the context of separate convexity in $\mathbb{R}^3$ in [27].

Next, for $1 < p < \infty$ let

$$k = 1 - \frac{2}{p},$$

so that $p = \frac{2}{1-k}$. We need to differentiate between the cases $1 < p \leq 2$ and $2 < p < \infty$.

The case $1 < p \leq 2$.

Let $\mu_N \in \mathcal{P}(\mathbb{R}^{2\times 2})$ be defined by the RHS of (3.7), more precisely

$$\int f\, d\mu_N := \frac{1}{1-k}\int_1^N \left[f\big(\mathrm{diag}(kt,t)\big) + f\big(\mathrm{diag}(t,kt)\big)\right] t^{-p}\,\frac{dt}{t} + \frac{f\big(\mathrm{diag}(N,N)\big)}{N^p}$$

for $f \in C(\mathbb{R}^{2\times 2})$. Then μ_N is a probability with barycenter $\overline{\mu}_N = \mathrm{diag}(1,1)$. Moreover, observe that if f is rank-one convex, then $(x,y) \mapsto f(\mathrm{diag}(x,y))$ is biconvex. Therefore, using Lemma 74 we see that μ_N is a laminate. Then, combining with the measure from Example 71 (c.f. splitting procedure from Definition 70) we conclude that the measure

$$\nu_N := \frac{1}{4}\mu_N + \frac{1}{4}\delta_{\mathrm{diag}(-1,1)} + \frac{1}{2}\delta_{\mathrm{diag}(0,-1)}$$

is a laminate with barycenter $\overline{\nu}_N = 0$. We claim that this sequence of laminates has the desired properties for Theorem 73. To this end we calculate

$$\int \phi_1\, d\mu_N = p|(1-k)^2 + \tau^2(1+k)^2|^{p/2}\log N + 2^p,$$
$$\int \phi_2\, d\mu_N = p(1+k)^p \log N.$$

In particular we see that as $N \to \infty$

$$\begin{aligned}
\frac{\int \phi_1 \, d\nu_N}{\int \phi_2 \, d\nu_N} \quad &\to \quad \frac{|(1-k)^2 + \tau^2(1+k)^2|^{p/2}}{(1+k)^p} \\
&= \left[\left(\frac{1-k}{1+k} \right)^2 + \tau^2 \right]^p \\
&= \left[\left(\frac{1}{p-1} \right)^2 + \tau^2 \right]^p = [(p^* - 1)^2 + \tau^2]^p.
\end{aligned}$$

The case $2 < p < \infty$.

Let $\tilde{\mu}_N \in \mathcal{P}(\mathbb{R}^{2\times 2})$ be defined by

$$\int f \, d\tilde{\mu}_N := \frac{1}{1-k} \int_1^N \big[f\big(\mathrm{diag}(-kt, t)\big) + f\big(\mathrm{diag}(-t, kt)\big)\big] t^{-p} \, \frac{dt}{t} + \frac{f\big(\mathrm{diag}(-N, N)\big)}{N^p}$$

for $f \in C(\mathbb{R}^{2\times 2})$. Then $\tilde{\mu}_N$ is a probability with barycenter $\overline{\tilde{\mu}}_N = \mathrm{diag}(-1, 1)$. Moreover, as before, we see that if f is rank-one convex, then $(x, y) \mapsto f(\mathrm{diag}(-x, y))$ is biconvex. Therefore $\tilde{\mu}_N$ is again a laminate, hence also

$$\tilde{\nu}_N := \frac{1}{4}\tilde{\mu}_N + \frac{1}{4}\delta_{\mathrm{diag}(1,1)} + \frac{1}{2}\delta_{\mathrm{diag}(0,-1)}$$

is a laminate with barycenter 0. Repeating the calculations above, we obtain

$$\begin{aligned}
\frac{\int \phi_1 \, d\tilde{\nu}_N}{\int \phi_2 \, d\tilde{\nu}_N} \quad &\underset{N\to\infty}{\longrightarrow} \quad \frac{|(1+k)^2 + \tau^2(1-k)^2|^{p/2}}{(1-k)^p} \\
&= \left[\left(\frac{1+k}{1-k} \right)^2 + \tau^2 \right]^p \\
&= [(p-1)^2 + \tau^2]^p = [(p^* - 1)^2 + \tau^2]^p.
\end{aligned}$$

3.3 Comparison with Burkholder functions

Now we will discuss the Burkholder functions introduced Chapter 2. Let $p^* - 1 := \max\left\{p-1, \frac{1}{p-1}\right\}$ and $x := (x_1, x_2)$ denote a point in $\mathbb{R}^2$. We will denote the coordinates $y := (y_1, y_2) \in \mathbb{R}^2$, as the rotation of x by $\frac{\pi}{4}$, that is

$$y_1 = \frac{x_1 + x_2}{2},\ y_2 = \frac{x_1 - x_2}{2}.$$

Definition 75. We say that a function $f := f(x_1, x_2)$ is *zigzag concave* if it is bi-concave in the y-variables.

Definition 76. Let $v(x_1, x_2) := v_{p,\tau}(x_1, x_2) = (\tau^2|x_1|^2 + |x_2|^2)^{\frac{p}{2}} - ((p^*-1)^2 + \tau^2)^{\frac{p}{2}}|x_1|^p$ and $u(x_1, x_2) := u_{p,\tau}(x_1, x_2) = \alpha_p(|x_1| + |x_2|)^{p-1}[|x_2| - (p^*-1)|x_1|]$, where $\alpha_p = p(1 - \frac{1}{p^*})^{p-1}\left(1 + \frac{\tau^2}{(p^*-1)^2}\right)^{\frac{p-2}{2}}$. For $1 < p < 2$, we define

$$U(x_1, x_2) := U_{p,\tau}(x_1, x_2) = \begin{cases} v(x_1, x_2) & : |x_2| \geq (p^*-1)|x_1| \\ u(x_1, x_2) & : |x_2| \leq (p^*-1)|x_1|, \end{cases}$$

and for $2 < p < \infty$,

$$U(x_1, x_2) := U_{p,\tau}(x_1, x_2) = \begin{cases} u(x_1, x_2) & : |x_2| \geq (p^*-1)|x_1| \\ v(x_1, x_2) & : |x_2| \leq (p^*-1)|x_1|. \end{cases}$$

Definition 77. Denote

$$c_{\mathcal{M}} := \inf\left\{c : v_c \text{ has a zigzag concave majorant and } U \text{ is such that } U(0,0) = 0\right\}.$$

Now we will see the key relationship between the Burkholder functions U and v.

Theorem 78. 1) $c_{\mathcal{M}} \geq ((p^* - 1)^2 + \tau^2)^{\frac{p}{2}}$, *for all* $1 < p < \infty$ *and all* $\tau \in \mathbb{R}$.
2) $c_{\mathcal{M}} = ((p^* - 1)^2 + \tau^2)^{\frac{p}{2}}$, *for* $1 < p < 2$ *and* $\tau^2 \leq \frac{1}{2p-1}$ *or* $2 \leq p < \infty$ *and* $\tau \in \mathbb{R}$.
3) *If* $1 < p < 2$ *and* τ *is sufficiently large then* $c_{\mathcal{M}} > ((p^* - 1)^2 + \tau^2)^{\frac{p}{2}}$.

Proof. 1) By way of contradiction, suppose that there is such a $\widetilde{c} \in [0, ((p^* - 1)^2 + \tau^2)^{\frac{p}{2}})$ that $v_{\widetilde{c}}$ has a zigzag concave majorant. Then following the upper bound estimate in Section 3.4, Theorem 93 would have $\widetilde{c}$ as the upper bound of our quadratic perturbation. However, this is impossible because of Theorem 73.

2) In Proposition 40 it was shown for $1 < p < 2$ and $\tau^2 \leq \frac{1}{2p-1}$ or $2 \leq p < \infty$ and $\tau \in \mathbb{R}$ that for $c = ((p^* - 1)^2 + \tau^2)^{\frac{p}{2}}$, v_c has a zigzag concave majorant. This proves that $c_{\mathcal{M}} \leq ((p^* - 1)^2 + \tau^2)^{\frac{p}{2}}$. Combining with 1) we get equality.

3) For $1 < p < 2$ and $\tau \in \mathbb{R}$ sufficiently large, U is no longer zigzag concave, while still being a majorant of v_c, with $c = ((p^* - 1)^2 + \tau^2)^{\frac{p}{2}}$. We know that if τ is sufficiently large and $1 < p < 2$, the least c_0 for which v_{c_0} has a zigzag concave majorant must satisfy $c_0 > ((p^* - 1)^2 + \tau^2)^{\frac{p}{2}}$. See [8], Remark 27. The condition $\tau^2 \leq \frac{1}{2p-1}$ is a sufficient condition for U to be the zigzag concave majorant, but not necessary. $\square$

3.3.1 Analyzing the Burkholder functions U and v

We will use the y-coordinates, unless otherwise stated, from this point on. In the y-coordinates,

$$\widetilde{U}(y_1, y_2) := U(y_1 - y_2, y_1 + y_2) = U(x_1, x_2),$$

and it takes the following form. For $2 \leq p < \infty$,

$$\widetilde{U}(y) = \begin{cases} \widetilde{u}(y), & \frac{p-2}{p} y_1 \leq y_2 \leq \frac{p}{p-2} y_1 \\ \widetilde{v}(y), & \text{otherwise,} \end{cases}$$

and for $1 < p < 2$,

$$\widetilde{U}(y) = \begin{cases} \widetilde{v}(y), & \frac{2-p}{p} y_1 \leq y_2 \leq \frac{p}{2-p} y_1 \\ \widetilde{u}(y), & \text{otherwise,} \end{cases} \tag{3.11}$$

where

$$\widetilde{u}(y_1, y_2) := u(y_1 - y_2, y_1 + y_2) = u(x_1, x_2), \widetilde{v}(y_1, y_2) := v(y_1 - y_2, y_1 + y_2) = v(x_1, x_2).$$

We will fix $2 \leq p < \infty$, as the dual range of p values is handled similarly. Denote

$$k := \frac{p}{p-2}. \tag{3.12}$$

Then $p = \frac{2k}{k-1}$ and $p - 1 = \frac{k+1}{k-1}$. Also denote

$$L_k := \{y_2 = k y_1\} \text{and } L_{\frac{1}{k}} := \left\{ y_2 = \frac{1}{k} y_1 \right\}. \tag{3.13}$$

Observe that in the cone

$$C_1 = \{y_1 \leq y_2 \leq k y_1\},$$

$\widetilde{U}$ is linear if we fix y_2. Also, $\widetilde{U}$ is linear if we fix y_1 in the cone

$$C_2 = \left\{ \frac{1}{k} y_1 \leq y_2 \leq y_1 \right\}.$$

Consequently, $\widetilde{U}$ is almost linear in the "T-shape" graph, which we will denote as T, with vertices

$$\left\{(\frac{1}{k}(y_1+h), y_1+h), (y_1, y_1+h), (y_1+h, y_1+h), (y_1, \frac{1}{k}y_1)\right\}.$$

The only portion where $\widetilde{U}$ is not linear is on the segment from (y_1, y_1) to (y_1, y_1+h). It is very small in comparison with the graph T.

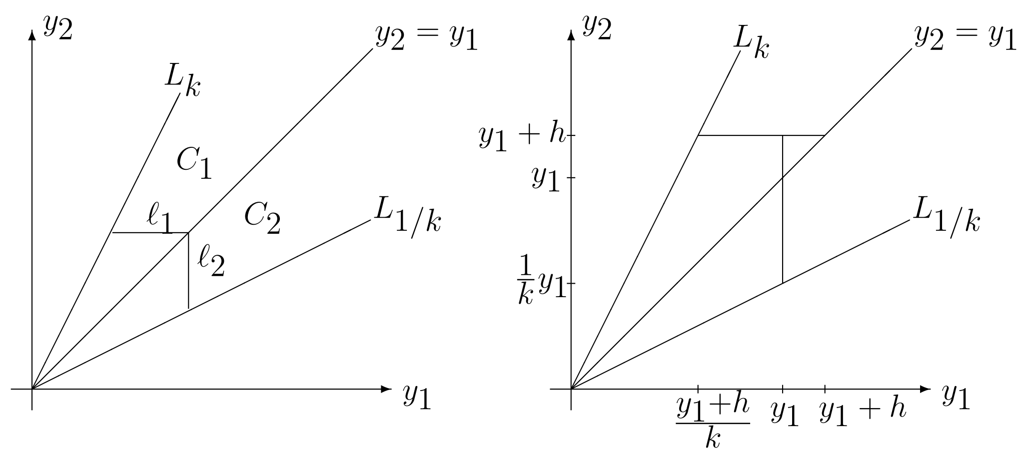

Figure 3.1: Splitting between u and $\widetilde{v}$ in $y_1 \times y_2$-plane.

By Theorem 78, $\widetilde{U} \geq \widetilde{v}$ in $\mathbb{R}^2$. But, on L_k and $L_{\frac{1}{k}}$, $\widetilde{U} = \widetilde{v}$. Also, observe that $\widetilde{U}(0,0) = \widetilde{v}(0,0)$ and $\widetilde{v} \geq 0$ on C_1 and C_2. (This is easy to see in x-coordinates.) We will summarize these important facts, so that we can later refer to them.

Proposition 79. *1) $\widetilde{v} \geq 0$ on C_1 and C_2*

2) $\widetilde{v}(0,0) = \widetilde{U}(0,0) = 0$

3) $\widetilde{U} = \widetilde{v}$ on L_k and $L_{\frac{1}{k}}$.

4) $\widetilde{U}$ is nearly linear on T.

3.3.2 Why the laminate sequence ν_N worked in Theorem 73

Let $\phi_1(y_1, y_2) := (|y_1 - y_2|^2 + \tau^2|y_1 + y_2|^2)^{\frac{p}{2}}$, $\phi_2(y_1, y_2) := |y_1 + y_2|^p$.

Definition 80. Let $c_{\mathcal{L}} := \sup\left\{\frac{\int \phi_1\, d\nu}{\int \phi_2\, d\nu} : \nu \in \mathcal{L}_0\right\}$.

Theorem 81. $c_{\mathcal{M}} \geq c_{\mathcal{L}}$.

Proof. By the definition of $c_{\mathcal{L}}$, there exists a laminate $\nu \in \mathcal{L}$ with barycenter 0, such that $\frac{\int \phi_1 d\nu}{\int \phi_2\, d\nu} > c_{\mathcal{L}} - \varepsilon$. This is equivalent to

$$\int [\phi_1 - (c_{\mathcal{L}} - \varepsilon)\phi_2] d\nu > 0. \tag{3.14}$$

We will now show that $\phi_1 - (c_{\mathcal{L}} - \varepsilon)\phi_2$ does not have a biconcave majorant. By changing the variables back to x_1, x_2 we see that this means that $v_{c_{\mathcal{L}}-\varepsilon}(x_1, x_2)$ would not have a zigzag concave majorant, thus proving that $c_{\mathcal{L}} - \varepsilon \leq c_{\mathcal{M}}$. By way of contradiction, suppose that $b := c_{\mathcal{L}} - \varepsilon$, and that $\phi_1 - b\phi_2 \leq U$, which is biconcave. Then by (3.14),

$$0 < \int (\phi_1 - b\phi_2) d\nu \leq \int U d\nu \leq U(\overline{\nu}) = U(0,0) = 0.$$

This gives a contradiction and we are finished with the proof. $\square$

Definition 82. We denote $(p, \tau) \in \mathcal{T}$, if $1 < p < 2$ and $\tau^2 \leq \frac{1}{2p-1}$ or $2 \leq p < \infty$ and $\tau \in \mathbb{R}$.

Let us compose $v_c(x_1, x_2)$ with the change of variables

$$x_1 = y_1 + y_2$$

$$x_2 = y_1 - y_2.$$

We get the function called

$$\widetilde{v}_c(y_1, y_2) := (|y_1 - y_2|^2 + \tau^2 |y_1 + y_2|^2)^{\frac{p}{2}} - c|y_1 + y_2|^p.$$

Recall that similarly $\widetilde{U}(y_1, y_2) := U(y_1 + y_2, y_1 - y_2) = U(x_1, x_2)$.

Let us introduce the following notation.

Definition 83. $c_{\mathcal{B}} := \left((p^* - 1)^2 + \tau^2\right)^{\frac{p}{2}}$.

Here $\mathcal{B}$ stands for Burkholder. Recall Definition 77 and let us also denote

Definition 84. $c_{\mathcal{N}} := \left\|\left(R_1^2 - R_2^2, \tau I\right)\right\|_{L^p(\mathbb{C},\mathbb{C}) \to L^p(\mathbb{C},\mathbb{C}^2)}$.

Remark 85. For $(p, \tau) \in \mathcal{T}, c_{\mathcal{L}} = c_{\mathcal{M}} = c_{\mathcal{B}}$.

This follows immediately from Theorems 73, 78 and 81.

Moreover, we saw that for all p, τ

$$c_{\mathcal{L}} \leq c_{\mathcal{N}}.$$

This is Corollary 72 essentially.

In Section 3.4 we are proving that

$$c_{\mathcal{N}} \leq c_{\mathcal{M}}.$$

We wish to discuss the set Ω of pairs (p, τ), for which $c_{\mathcal{L}} = c_{\mathcal{M}}$. This set of pairs contains $\mathcal{T}$ introduced above, but we do not know exactly the whole Ω. By what we have just said if $(p, \tau) \in \Omega$, then the norm of our operator is $c_{\mathcal{L}}$, and we also know that for such pairs the sharp estimate from above for the norm is obtainable by means of finding the least c for which v_c has a zigzag concave majorant.

Now we want to see what kind of restriction the equality $c_{\mathcal{L}} = c_{\mathcal{M}}$ imposes *a priori*.

Conjecture 86. *If* p, τ *are such that* $c_{\mathcal{L}} = c_{\mathcal{M}}$*, then there exists a sequence of laminates* $\{\nu_N\}$ *with barycenter* 0*, such that* $\int \widetilde{U}c_{\mathcal{M}} d\nu_N$ *increases to* 0*.*

Remark 87. In fact, this is exactly what happens on $\mathcal{T}$. Namely, if $\nu_N = \frac{1}{4}\mu_N + \frac{1}{4}\delta_{(-1,1)} + \frac{1}{2}\delta_{(0,-1)}$ and $(p,\tau) \in \mathcal{T}$, then

$$-\mathcal{O}(1) \leq \int \widetilde{U}c_{\mathcal{M}} d\nu_N \leq 0.$$

If (p,τ) is not in $\mathcal{T}$, but $c_{\mathcal{L}} = c_{\mathcal{M}}$, then we get something interesting. By the definition of $c_{\mathcal{L}}$, there exists some ν_N with barycenter 0, and such that $\frac{\int \phi_1\, d\nu_N}{\int \phi_2\, d\nu_N} \geq c_{\mathcal{M}} - \frac{1}{N}$. Then we get

$$-\frac{1}{N} \leq \frac{\int(\phi_1 - c_{\mathcal{M}}\phi_2) d\nu_N}{\int \phi_2 d\nu_N} \leq \frac{\int \widetilde{U}c_{\mathcal{M}} d\nu_N}{\int \phi_2 d\nu_N} \leq 0.$$

Therefore,

$$\Big|\int \widetilde{U}c_{\mathcal{M}}\, d\nu_N\Big| = o\Big(\int \phi_2\, d\nu_N\Big). \tag{3.15}$$

For (3.15) it would be sufficient to have

$$\Big|\int \widetilde{U}c_{\mathcal{M}}\, d\nu_N\Big| = \mathcal{O}(1).$$

Remembering that $\int \widetilde{U}c_{\mathcal{M}}\, d\nu_N \leq 0$ we can write this as

$$-c \leq \int \widetilde{U}c_{\mathcal{M}} d\nu_N \leq 0.$$

This was exactly the case for $(p,\tau) \in \mathcal{T}$ with $\nu_N = \frac{1}{4}\mu_N + \frac{1}{4}\delta_{(-1,1)} + \frac{1}{2}\delta_{(0,-1)}$. The reason for that was because

$$\overline{\mu}_N = (1,1) \text{ and } \int \widetilde{U}c_{\mathcal{M}} d\mu_N = \widetilde{U}c_{\mathcal{M}}(1,1) \tag{3.16}$$

For any w biconcave and μ with barycenter $(1,1)$ we have $\int w\, d\mu \leq w(1,1)$. But in (3.16) we have the case when the equality is attained. To understand better the case when the equality can be attained when integrating a biconcave function against a laminate, let us consider first a simpler question when equality is attained in integrating the usual concave function against a usual probability measure.

If w is concave, then $\int w\, d\mu \leq w(1,1)$ is true for any probability measure μ. There are only two ways to get equality (i.e., $\int w\, d\mu = w(1,1)$): 1) if μ is a delta measure at $(1,1)$ or 2) if w is linear on the convex hull of the support of measure μ (degenerate concave).

Coming back to the attained equality in (3.16), for biconcave $\widetilde{U}c_{\mathcal{M}}$, we see that (3.16) happened also exactly because the Burkholder function $\widetilde{U}c_{\mathcal{M}}$, is not only biconcave on the cones $C_1 \cup C_2$, but degenerate biconcave, meaning that $C_1 \cup C_2$ is foliated by curves on which one of the concavities degenerates into linearity. We may conjecture that the same geometric picture happens for those (p,τ) outside of $\mathcal{T}$, for which $c_{\mathcal{L}} = c_{\mathcal{M}}$, but we do not know how to prove this.

To summarize, we have the following.

- For all p and $\tau, c_{\mathcal{M}} \geq c_{\mathcal{N}} \geq c_{\mathcal{L}} \geq \left((p^*-1)^2 + \tau^2\right)^{\frac{p}{2}} =: c_{\mathcal{B}}(p,\tau)$. All four constants coincide at least for $(p,\tau) \in \mathcal{T}$.

- For all $p \in (1,2)$, there exists a τ_0 such that for all $|\tau| > \tau_0, c_{\mathcal{M}} > c_{\mathcal{B}}(p,\tau)$ (by [7]

Remark 27).

- (p, τ) such that $c_{\mathcal{L}} = c_{\mathcal{M}}$ holds for all $(p, \tau) \in \mathcal{T}$, but may also be true outside of $\mathcal{T}$.

- By a modification of [7], Remark 27, one can prove that for all $p \in (1, 2)$, there exits τ_0 such that, for all $|\tau| \geq \tau_0, c_{\mathcal{N}} > c_{\mathcal{B}}(p, \tau)$.

- We of course expect that always $c_{\mathcal{N}} = c_{\mathcal{L}}$.

3.4 Upper Bound Estimate

3.4.1 Background information and notation

We will use similar notation, estimates and reasoning developed in [2] and [3]. Let $B_t = (Z_t, T - t)$ denote space-time Brownian motion starting at $(0, T) \in \mathbb{R}^3_+ := \mathbb{R}^2 \times (0, \infty)$, where Z_t is standard Brownian motion in the plane. There is a pseudo-probability measure P^T associated with the process and we will denote $\mathbb{E}^T$ as the corresponding expectation.

For $\phi \in C_c^\infty(\mathbb{C})$, we denote $U_\phi(z, t)$, as the heat extension to the upper half space, in other words U_ϕ is the solution to

$$\begin{cases} \partial_t U_\phi - \frac{1}{2}\Delta U_\phi, & R^3_+ \\ U_\phi = \phi, & R^2. \end{cases}$$

By Itô's formula we get the relation,

$$U_\phi(B_t) - U_\phi(B_0) = \int_0^t \nabla U_\phi(B_s) \cdot dZ_s, \tag{3.17}$$

which is a martingale. For a 2×2 matrix A we denote

$$(A * U_\phi)_t := \int_0^t A \nabla U_\phi(B_s) \cdot dZ_s$$

as a martingale transform. Throughout Section 3.4 we will refer to the matrices

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

If we rewrite (3.17) in the form

$$X_t = X_1^t + iX_2^t = (I * \phi)_t = \int_0^t \nabla U_\phi(B_s) \cdot dZ_s,$$

then its martingale transform will be denoted as

$$Y_t = Y_1^t + iY_2^t = ((A_1 - A_2) * \phi)_t = \int_0^t (A_1 - A_2) \nabla U_\phi(B_s) \cdot dZ_s.$$

The quadratic variation of X_i and Y_i are

$$\langle X_i \rangle_t = \int_0^t |\nabla U_{\phi_i}(B_s)|^2\, ds = \int_0^t \left| \begin{pmatrix} \partial_x U_{\phi_i}(B_s) \\ -\partial_y U_{\phi_i}(B_s) \end{pmatrix} \right|^2 ds = \langle Y_i \rangle_t, \text{ for } i = 1, 2.$$

Then, $\langle X \rangle_t = \langle X_1 \rangle_t + \langle X_2 \rangle_t = \langle Y_1 \rangle_t + \langle Y_2 \rangle_t = \langle Y \rangle_t$.

Definition 88. A process H is called differentially subordinate to a process K, if $\frac{d}{dt}\langle H \rangle_t \leq \frac{d}{dt}\langle K \rangle_t$.

We have computed that Y is differentially subordinate to X. Note that Y is the continuous version of the martingale transform (the discrete version of Burkholder's martingale transform is $\sum_{k=1}^{n} d_k \to \sum_{k=1}^{n} \varepsilon_k d_k$, where $\varepsilon_k \in \{\pm 1\}$ and $\{d_k\}_k$ is a martingale difference sequence and $n \in \mathbb{Z}_+$), since Y is differentially subordinate to X.

3.4.2 Extending the martingale estimate to continuous time martingales

Theorem 89. *Let X and Y be two complex-valued martingales, such that Y is the martingale transform of X (in other words $\frac{d}{dx}\langle X\rangle_t \le \frac{d}{dx}\langle Y\rangle_t$). Then*

$$\|\tau^2|X|^2 + |Y|^2\|_p \le \big((p^* - 1)^2 + \tau^2\big)^{\frac{1}{2}} \|X\|_p,$$

with the best possible constant for $1 < p < 2$ and $\tau^2 \le \frac{1}{2p-1}$ or $2 \le p < \infty$ and $\tau \in \mathbb{R}$.

This was basically shown in Chapter 2, but we will give the idea of the proof. The proof here only requires the same modification to continuous time martingales as was done in [3], for $\tau = 0$. Let

$$u(x,y) = p\Big(1 - \frac{1}{p^*}\Big)^{p-1}\Big(1 + \frac{\tau^2}{(p^*-1)^2}\Big)^{p-2}(|y| - (p^*-1)|x|)(|x| + |y|)^{p-1} \text{ and}$$
$$v(x,y) = (\tau^2|x|^2 + |y|^2)^{\frac{p}{2}} - \big((p^*-1)^2 + \tau^2\big)^{\frac{1}{2}}|x|^p.$$

It was shown in Theorem 78 that $v \le u$. The key properties of u and v, that will be used, are:

(1) $v(x,y) \le u(x,y)$

(2) For all $x, y, h, k \in \mathbb{C}$, if $|x||y| \neq 0$, then

$$\langle hu_{xx}(x,y), h\rangle + 2\langle hu_{xy}(x,y), k\rangle + \langle ku_{yy}(x,y), k\rangle = -c_{p,\tau}(A+B+C),$$

where $c_{p,\tau} > 0$ is a constant only depending on τ and p and

$$A = p(p-1)(|h|^2 - |k|^2)(|x|+|y|)^{p-2}, \; B = p(p-2)[|k|^2 - (y', k)^2]|y|^{-1}(|x|+|y|)^{p-1},$$
$$C = p(p-1)(p-2)[(x', h) + (y', k)]^2|x|(|x|+|y|)^{p-3},$$

where $x' = x/|x|$, $y' = y/|y|$.

(3) $u(x,y) \leq 0$ if $|y| \leq |x|$.

Since u here only differs from the one in [3] by a multiple of $\left(1+\frac{\tau^2}{(p^*-1)^2}\right)^{p-2}$, the rest of the argument follows in an identical way which we briefly outline.

By Itô's formula,

$$u(X_t, Y_t) = u(X_0, Y_0) + \int_0^t \langle u_x(X_s, Y_s), dX_s\rangle + \int_0^t \langle u_y(X_s, Y_s), dY_s\rangle + \frac{I_t}{2},$$

where I_t contains the second order terms. We can assume, without loss of generality that $|Y_0| \leq |X_0|$, so that when we take expectation of $u(X_t, Y_t)$, we obtain $\mathbb{E}u(X_t, Y_t) \leq \mathbb{E}(I_t/2)$. Using property (2) above, in the martingale setting, one can obtain

$$I_t \leq -c_{p,\tau} \int_0^t (|X_s| + |Y_s|)^{p-2} d(\langle X\rangle_s - \langle Y\rangle_s) \leq 0,$$

since $B, C \geq 0$ and using the differential subordinate assumption. Therefore, $\mathbb{E}v(X_t, Y_t) \leq 0$ by property (1) above.

3.4.3 Connecting the martingales to the Riesz transforms

Now we choose $X_t := (I * U_\phi)_t$ and $Y_t := ((A_1 - A_2) * U_\phi)_t$ to obtain the following corollary of Theorem 89.

Corollary 90. *If* $1 < p < 2$ *and* $\tau^2 \leq \frac{1}{2p-1}$ *or* $2 \leq p < \infty$ *and* $\tau \in \mathbb{R}$*, then*

$$\|\tau^2 |(I * U_\phi)_t|^2 + |((A_1 - A_2) * U_\phi)_t|^2\|_p \leq ((p^* - 1)^2 + \tau^2)^{\frac{1}{2}} \|(I * U_\phi)_t\|_p.$$

Proposition 91. *For all* $\phi \in C_c^\infty$ *and all* $p \in (1, \infty)$*,* $\lim_{T\to\infty} \|(I * U_\phi)_T\|_p \leq \|\phi\|_p$.

This result was proven in [2].

Now we will connect the martingales X_t and Y_t with the planar Riesz transforms,R_1 and R_2, in the following way.

Proposition 92. *For all* $\phi \in C_c^\infty(\mathbb{C})$,

$$\begin{aligned}
&\lim_{T\to\infty} \int_{\mathbb{C}} \left[|\mathbb{E}^T(Y_T|B_T = z)|^2 + \tau^2 |\mathbb{E}^T(X_T|B_T = z)|^2\right]^{\frac{p}{2}} dz \\
&= \int_{\mathbb{C}} \left[|(R_1 - R_2^2)\phi|^2 + \tau^2 |(R_1 + R_2^2)\phi|^2\right]^{\frac{p}{2}} dz.
\end{aligned}$$

This result follows almost immediately from the fact that, for all $\psi, \phi \in C_c^\infty(\mathbb{C})$,

$$\lim_{T\to\infty} \int_{\mathbb{C}} \psi \mathbb{E}^T[Y_T|B_T = z]\, dz = \int_{\mathbb{C}} \psi (R_1^2 - R_2^2)\phi\, dz \text{ and} \tag{3.18}$$

$$\lim_{T\to\infty} \int_{\mathbb{C}} \psi \mathbb{E}^T[X_T|B_T = z]\, dz = \int_{\mathbb{C}} \psi (R_1^2 + R_2^2)\phi\, dz, \tag{3.19}$$

by [2]. By (3.18) and (3.19) we obtain that for all $\psi, \phi \in C_c^\infty(\mathbb{C})$,

$$\lim_{T\to\infty} \int_{\mathbb{C}} \psi [|\mathbb{E}^T(Y_T|B_T = z)|^2 + \tau^2 |\mathbb{E}^T(X_T|B_T = z)|^2]^{\frac{1}{2}}\, dz$$
$$= \int_{\mathbb{C}} \psi [|(R_1 - R_2^2)\phi|^2 + \tau^2 |(R_1 + R_2^2)\phi|^2]^{\frac{1}{2}}\, dz.$$

3.4.4 Main Result

Theorem 93. *For* $1 < p < 2$ *and* $\tau^2 \le \frac{1}{2p-1}$ *or* $2 \le p < \infty$ *and* $\tau \in \mathbb{R}$ *we have the following estimate.*

$$\left\| [|(R_1^2 - R_2^2)f|^2 + \tau^2 |f|^2]^{\frac{1}{2}} \right\|_p \le ((p^* - 1)^2 + \tau^2)^{\frac{1}{2}} \|f\|_p$$

Let $\mathbb{E}^{z_0,T}$ correspond to Brownian motion starting at $(z_0, T) \in \mathbb{R}^3_+$. Let $\phi \ge 0$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\int_{\mathbb{C}} (|(R_1^2 - R_2^2)\phi|^2 + \tau^2 |\phi|^2)^{\frac{1}{2}} \psi(z) dz$$
$$= \int_{\mathbb{C}} \lim_{T\to\infty} \int_{\mathbb{C}} (|\mathbb{E}^{(z_0,T)}(Y_T|B_T = z)|^2 + \tau^2 |\mathbb{E}^{(z_0,T)}(X_T|B_T = z)|^2)^{\frac{1}{2}} dz_0 \psi(z) dz$$
$$= \lim_{T\to\infty} \int_{\mathbb{C}} \int_{\mathbb{C}} \left| \begin{pmatrix} \mathbb{E}^{(z_0,T)}(Y_T \psi(Z_T)|B_T = z) \\ \tau \mathbb{E}^{(z_0,T)}(X_T \psi(Z_T)|B_T = z) \end{pmatrix} \right| dz\, dz_0$$
$$\le \lim_{T\to\infty} \int_{\mathbb{C}} \mathbb{E}^{(z_0,T)} \left| \begin{pmatrix} Y_T \psi(Z_T) \\ \tau X_T \psi(Z_T) \end{pmatrix} \right| dz_0$$
$$\le \left(\lim_{T\to\infty} \int_{\mathbb{C}} \mathbb{E}^{(z_0,T)} \left| \begin{pmatrix} Y_T \\ \tau X_T \end{pmatrix} \right|^p dz_0 \right)^{\frac{1}{p}} \left(\lim_{T\to\infty} \int_{\mathbb{C}} \mathbb{E}^{(z_0,T)} |\phi(Z_T)|^q dz_0 \right)^{\frac{1}{q}}$$

$$=\left(\lim_{T\to\infty}\int_{\mathbb{C}}\mathbb{E}^{(z_0,T)}\left|\begin{pmatrix} Y_T \\ \tau X_T \end{pmatrix}\right|^p dz_0\right)^{\frac{1}{p}}\|\psi\|_{L^q},$$

where the last equality is by Proposition 91. By linearity we have this result for any $\psi \in L^q$. Therefore, by duality

$$\left(\int_{\mathbb{C}}(|(R_1^2-R_2^2)\phi|^2+\tau^2|\phi|^2)^{\frac{p}{2}}\right)^{\frac{1}{p}} \leq \left(\lim_{T\to\infty}\int_{\mathbb{C}}\mathbb{E}^{(z_0,T)}\left|\begin{pmatrix} Y_T \\ \tau X_T \end{pmatrix}\right|^p dz_0\right)^{\frac{1}{p}}$$
$$=(\lim_{T\to\infty}\mathbb{E}^T[(|Y_T|+\tau^2|X_T|^2)^{\frac{p}{2}}])^{\frac{1}{p}} \leq ((p^*-1)^2+\tau^2)^{\frac{1}{2}}\lim_{T\to\infty}(\mathbb{E}|X_T|^p)^{\frac{1}{p}}$$
$$=((p^*-1)^2+\tau^2)^{\frac{1}{2}}\|\phi\|_{L^p},$$

where the last inequality is due to Theorem 89 and the last equality is by Proposition 92.

Corollary 94. *For $1 < p < 2$ and $\tau^2 \leq \frac{1}{2p-1}$ or $2 \leq p < \infty$ and $\tau \in \mathbb{R}$,*

$$\begin{aligned}\|(2R_1R_2\,,\,\tau I)\|_{L^p(\mathbb{C},\mathbb{C})\to L^p(\mathbb{C},\mathbb{C}^2)} &= \left\|\left(R_1^2-R_2^2\,,\,\tau I\right)\right\|_{L^p(\mathbb{C},\mathbb{C})\to L^p(\mathbb{C},\mathbb{C}^2)} \\ &= \left((p^*-1)^2+\tau^2\right)^{\frac{1}{2}}.\end{aligned}$$

Since $R_1^2-R_2^2$ and $2R_1R_2$ are just rotations of one another by $\pi/4$, we have the equality of the two operator norms. The lower bound was computed as $\left((p^*-1)^2+\tau^2\right)^{\frac{1}{2}}$ in Theorem 73 (or by another technique in [10]). The upper bound was just computed as the same, giving the desired result.

Chapter 4

Dissertation achievements and future work

4.1 Contributions

A large class of Fourier multiplier operators, can be built out of linear combinations of the product of two Riesz transforms. The Ahlfors–Beurling operator is one such example, which has been discussed extensively in this dissertation. Fourier multiplier operators are also referred to as singular integrals, a designation which becomes clear by looking at the definition in (4.1). Let us now define what the Riesz transforms are.

The jth Riesz transform operator on $\mathbb{R}^d$ is defined as

$$R_j f(x) := c_d \lim_{\varepsilon \to 0} \int_{\mathbb{R}^d \setminus B(x,\varepsilon)} \frac{(x_j - t_j) f(t)}{|x-t|^{d+1}} \, dt \tag{4.1}$$

and is a well defined operator that takes L^p functions to L^p functions, where

$$c_d := \frac{\Gamma((d+1)/2)}{\pi^{(d+1)/2}}$$

is a constant which was suitably chosen so that (4.2) does not have any extra constants (and Γ is the well known "gamma function"). One of the key properties of this operator is that it is also a Fourier multiplier operator; in other words, if "$\widehat{\ }$" denotes the Fourier transform, then

$$\widehat{R_j f}(\xi) = i\frac{\xi_j}{|\xi|}\widehat{f}(\xi), \tag{4.2}$$

with $m_j(\xi) = i\frac{\xi_j}{|\xi|}$ being the associated Fourier multiplier. (Throughout $\widehat{g}$ denotes the Fourier transform of g and $g^{\vee}$ denotes the inverse Fourier transform.)

The Riesz transforms arise naturally in partial differential equations in the following way. If f is a rapidly decreasing function on R^d, and u is a solution to the Laplace equation,

$$\Delta u = f,$$

then taking the Fourier transform of the equation twice gives the well known formula $\widehat{u} = \frac{1}{|\xi|^2}\widehat{f}$. So we have

$$R_j R_k f = \left(-\frac{\xi_j \xi_k}{|\xi|^2}\widehat{f}\right)^{\vee} = (-\xi_j \xi_k \widehat{u})^{\vee} = -u_{y_j y_k}. \tag{4.3}$$

Note that, we are only looking at the product (composition) of two Riesz transforms, since that will be the focus here, but there is a similar formula for just one Riesz transform. So, these Riesz transforms naturally arise by taking partial derivatives of solutions of partial differential equations. A. Calderón and A. Zygmund showed in the 1950's that a large class of singular integrals, including the Fourier multiplier operators (as long as the corresponding multiplier is bounded), are bounded in L^p (meaning that their operator norm from L^p to L^p is finite). Knowing that a singular integral is bounded can help gain more information about the associated partial differential equation and its solution. But, being able to compute its exact operator norm can give even more information. One such singular integral whose L^p operator norm we computed exactly is a quadratic perturbation of the $R_1^2 - R_2^2$ (the real part of the Ahlfors–Beurling operator).

The approach of determining the operator norm of the perturbation of $R_1^2 - R_2^2$, was to first determine the operator norm of the same perturbation of the martingale transform. In Chapter 2, the exact L^p- operator norm was found for the quadratic perturbation of the martingale transform. Furthermore, we found the exact Bellman function and the Burkholder functions associated with the problem. Therefore, the celebrated result of Donald Burkholder proven in the series of papers, [12] to [19], has been generalized.

From the results of Chapter 2, there are several different approaches for determining the sharp lower bound estimate of the following operator norm: $\left\|\left(R_1^2 - R_2^2\,,\,\tau I\right)\right\|_{L^p \to L^p}$, where R_1 and R_2 are the two components of the Riesz transform in the plane. One approach uses a similar technique developed by J. Bourgain [11] which has then been generalized by Geiss, Montgomery-Smith, Saksman [24]. The sharp lower bound estimate can be found using this approach (in N. Boros, A. Volberg [10]), however we will instead discuss a new approach described in Chapter 3. This new approach does not use the estimates obtained in Theorem 43 of Chapter 2, but relies on the interaction between the Burkholder functions U and v used to obtain the sharp estimate. This is an entirely new approach of estimating singular integral operators from below, in which we constructed an almost extremal sequence for the operator, by means of laminates. The appropriate laminate sequence was constructed by carefully studying the relationships of the Burkholder functions.

It should also be noted that there are very few singular integral operators whose exact operator norm is known, as it is quite difficult to calculate this quantity for many singular integral operators. The astute reader of Chapters 2 and 3, can see that this truly is the case for $\left(R_1^2 - R_2^2, \tau I\right)$.

Most of the materials of this dissertation are adopted from the following publications and preprints:

- **N. Boros**, L. Szèkelyhidi, Jr., A. Volberg, "Laminates Meet Burkholder Functions." *submitted to the Duke Mathematical Journal*, (2011).

- **N. Boros**, P. Janakiraman, A. Volberg, "Perturbation of Burkholder's martingale transform and Monge-Ampère equation", *submitted to Advances in Mathematics Journal*, (2011).

- **N. Boros**, P. Janakiraman, A. Volberg, "Sharp L^p-bounds for a small perturbation of Burkholder's martingale transform", *to appear in Indiana University Mathematics Journal*, (2011).

- **N. Boros**, P. Janakiraman, A. Volberg, "Sharp L^p-bounds for a perturbation of Burkholders Martingale Transform", *Ser. I, C. R. Acad. Sci. Paris, Ser. I*, 349: 303–307, (2011).

- **N. Boros**, A. Volberg, "Sharp Lower bound estimates for vector-valued and matrix-valued multipliers in L^p", *arXiv:1110.5405v1*, (2010).

4.2 Future work

Let B denote the Ahlfors-Beurling operator and $\|B\|_p$ denote the operator norm from L^p to L^p. It was shown in 1965, by O. Lehto [28], that $\|B\|_{p \to p} \geq p^* - 1$. T. Iwaniec conjectured in 1982, see [25], that $\|B\|_{p \to p} = p^* - 1$. There have been many attempts at proving this conjecture: Bañuelos, Wang [4], Nazarov, Volberg [31], Bañuelos, Mèndez-Hernàndez [2], Dragičević, Volberg [22], Bañuelos, Janakiraman [3] is the current list of attempts, which have slowly gotten closer to the result but have still not attained it. The implications of the validity of this nearly 30 year old conjecture are broad reaching into several fields, such as harmonic analysis, partial differential equations and in the study of quasi-conformal mappings (for more details on the implications see [22]).

There are two difficulties in estimating B. One is that B is a complex operator, in fact there are very few complex operators for which we know the exact L^p operator norm. Secondly, B is a linear combination of squares of Riesz transforms and a product of two Riesz transforms. (Recall that $\Re B = R_1^2 - R_2^2$ and $\Im B = 2R_1R_2$.) The method developed and implemented for attaining the result in Corollary 94 can be applied to computing the exact L^p norm of many other perturbation operators. In fact, two other such perturbations are a linear perturbation already mentioned, $\Re B + \tau \cdot I$ and a complex perturbation $\Re B + \tau \cdot iI$. Since

$$\begin{aligned} aR_1^2 + bR_2^2 &= \frac{a-b}{2}(R_1^2 - R_2^2) + \frac{a+b}{2}(R_1^2 + R_2^2) \\ &= \frac{a-b}{2}\Big[(R_1^2 - R_2^2) + \frac{a+b}{a-b}(R_1^2 + R_2^2)\Big] \\ &= \frac{a-b}{2}\Big[(R_1^2 - R_2^2) + \frac{a+b}{a-b}\, I\Big], \end{aligned}$$

the linear perturbation is just a constant multiple of $aR_1^2 + bR_2^2$. So we now have a way for approaching the L^p operator norm of a linear combination of squares of Riesz transforms as well as for a complex operator. The two results would be interesting not just in obtaining the exact L^p of the perturbation of $\Re B$, but along the way two other very interesting generalizations of Burkholder's famous result would be found as well. These two problems are joint works with P. Janakiraman and A. Volberg.

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