Sharp L^p-bounds for a perturbation of Burkholder's Martingale Transform

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Abstract

Let $\{d_k\}_{k\geq 1}$ be a real martingale difference in $L^p[0,1)$, where $1<p<\infty$, and $\{\varepsilon_k\}_{k\geq 1} \subset \{\pm 1\}$. We obtain the following generalization of Burkholder's famous result. If $\tau \in [-\frac{1}{2}, \frac{1}{2}]$ and $n \in \mathbb{Z}_+$ then $\left\|\sum_{k=1}^n (\varepsilon_k d_k, \tau d_k)\right\|_{L^p([0,1),\mathbb{R}^2)} \leq ((p^*-1)^2+\tau^2)^{\frac{1}{2}} \left\|\sum_{k=1}^n d_k\right\|_{L^p([0,1),\mathbb{R})}$, where $((p^*-1)^2+\tau^2)^{\frac{1}{2}}$ is sharp and $p^*-1 = \max\{p-1, \frac{1}{p-1}\}$. *To cite this article: A. Nicholas Boros, A. Prabhu Janakiraman, Alexander Volberg, C. R. Acad. Sci. Paris, Ser. I .*

Résumé

Estimations optimales dans L_p pour des transformations de martingale perturbées. Le calcul de la norme des opérateurs du type intégrales singulières est connu seulement dans très peu de cas. Le cas le plus célèbre est celui de la transformation de martingale dans L^p trouvée par Burkholder égale à p^*-1. Outre des résultats de Pichorides [6], on peut signaler un résultat de Choi [3] et un calcul récent de $\|R_1^2 - R_2^2\|_p$ par Nazarov–Volberg [5], Banuelos–Janakiraman [1], et par Geiss–Montgomery-Smith–Saksman [4]. Les transformées de Riesz sur $\mathbb{R}^2$ sont notées R_1 et R_2. La note est consacrée à un résultat qui donne la norme d'une certaine perturbation de $R_1^2 - R_2^2$.

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égale à p^*-1. Ici on obtient une formule pour la norme d'une certaine perturbation de la transformation de martingale de Burkholder. Comme conséquence, on peut obtenir une formule précise pour la norme d'une petite perturbation de l'opérateur $R_1^2 - R_2^2$ qui joue un rôle important parce qu'il est la partie réelle de la transformation de Ahlfors–Beurling. Voici notre estimation d'une perturbation de la transformation de martingale : $\|\sum_{k=1}^n (\varepsilon_k d_k, \tau d_k)\|_{L^p([0,1),\mathbb{R}^2)} \le ((p^*-1)^2+\tau^2)^{\frac12} \|\sum_{k=1}^n d_k\|_{L^p([0,1),\mathbb{R})}$, où $((p^*-1)^2+\tau^2)^{\frac12}$ est optimale et $p^*-1 = \max\{p-1, \frac{1}{p-1}\}$. Cela est vrai pour tous les petits τ's. Et voici notre estimation de la perturbation de la partie réelle de l'opérateur de Ahlfors–Beurling (ici encore pour tous les petits τ's). :

$$\left\| \left(R_1^2 - R_2^2 \, , \, \tau I \right) \right\|_{L^p(\mathbb{C},\mathbb{R}) \to L^p(\mathbb{C},\mathbb{R}^2)} = ((p^*-1)^2 + \tau^2)^{\frac12} \, .$$

Notre résultat est valable pour les fonctions réelles. Pour des fonctions complexes, nous pouvons le démontrer pour $1 < p \le 2$ et pour tous les p pairs. Comme la preuve utilise la théorie des intégrales stochastiques on rencontre dans le cas général les mêmes difficultés relatives aux martingales complexes que l'on rencontre dans des tentatives pour résoudre un célèbre problème $p-1$ de Iwaniec. Néanmoins, nous pensons qu'ici on pourra géneraliser ce résultat au cas des fonctions complexes.

Notre approche est basée sur l'utilisation de l'équation de Monge–Ampère. Plus précisement, on commence par la recherche d'une fonction $M(y_1, y_2, y_3)$ définie dans $\Omega := \{(y_1, y_2, y_3) : y_3 \ge |y_1 - y_2|^p\}$ et telle que M est bi-concave par rapport à (y_1, y_3) et (y_2, y_3) et telle que sur $\partial\Omega$ on a $M = ((y_2+y_1)^2 + \tau^2 (y_1 - y_2)^2)^{p/2}$. L'équation de Monge–Ampère apparait parce que maintenant nous voulons trouver M satisfaisant cette équation dans les variables y_2, y_3 pour chaque y_1 fixé dans l'intersection de Ω avec $y_1 - y_2 > 0$, $y_1 + y_2 > 0$.

1. Introduction

Let I be an interval and $\alpha^\pm \in \mathbb{R}^+$ such that $\alpha^+ + \alpha^- = 1$. These $\alpha^\pm$ generate two subintervals $I^\pm$ such that $|I^\pm| = \alpha^\pm |I|$ and $I = I^- \cup I^+$. We can continue this decomposition indefinitely as follows. For any sequence $\{\alpha_{n,m} : 0 < \alpha_{n,m} < 1, 0 \le m < 2^n, 0 < n < \infty, \alpha_{n,2k} + \alpha_{n,2k+1} = 1\}$, we can generate the sequence $\mathcal{I} := \{I_{n,m} : 0 \le m < 2^n, 0 < n < \infty\}$, where $I_{n,m} = I^-_{n,m} \cup I^+_{n,m} = I_{n+1,2m+1} \cup I_{n+1,2m+1}$ and $\alpha^- = \alpha_{n+1,2m}, \alpha^+ = \alpha_{n+1,2m+1}$. Note that $I_{0,0} = I$. For any $J \in \mathcal{I}$ we define the Haar function $h_J := -\sqrt{\frac{\alpha^+}{\alpha^-|J|}}\chi_{J^-} + \sqrt{\frac{\alpha^-}{\alpha^+|J|}}\chi_{J^+}$.

Burkholder defined a martingale transform [2], MT_ε, as $MT_\varepsilon\left(\sum_{k=1}^n d_k\right) := \sum_{k=1}^n \varepsilon_k d_k$. Using this definition of the martingale transform, we can state one of the main results as

$$\left\| \left(MT_\varepsilon \left(\sum_{k=1}^n d_k \right) \, , \, \tau \left(\sum_{k=1}^n d_k \right) \right) \right\|_{L^p([0,1),\mathbb{R}^2)} \le ((p^*-1)^2 + \tau^2)^{\frac12} \left\| \sum_{k=1}^n d_k \right\|_{L^p([0,1),\mathbb{R})} , \tag{1}$$

where $\tau \in [-\frac12, \frac12]$. We will define the martingale transform slightly differently. Burkholder in [2] showed that the norm in L^p has to be the same for this more general definition. Fix $I = [0,1)$ and $\mathcal{I} = \mathcal{D}$ as the dyadic subintervals of I. Denote by $\langle f \rangle_J$ the average $\frac{1}{|J|}\int_J f, J \in \mathcal{D}$. Then $f \in L^p[0,1)$ can be decomposed in terms of the Haar system as $f = \langle f \rangle_{[0,1)}\chi_{[0,1)} + \sum_{I\in\mathcal{D}} (f, h_I) h_I$. We define the martingale transform, g of f, as

$$g := \langle g \rangle_{[0,1)}\chi_{[0,1)} + \sum_{I\in\mathcal{D}} \varepsilon_I (f, h_I) h_I ,$$

where $\varepsilon_I \in \{\pm 1\}$. Requiring that $|(g, h_J)| = |(f, h_J)|$ for all $J \in \mathcal{D}$ is equivalent to g being the martingale transform of f.

Now we define the Bellman function as

$$\mathcal{B}(x_1, x_2, x_3) := \sup_{f,g}\{\langle (g^2 + \tau^2 f^2)^{\frac{p}{2}}\rangle_I : x_1 = \langle f\rangle_I, x_2 = \langle g\rangle_I, x_3 = \langle |f|^p\rangle_I, |(f, h_J)| = |(g, h_J)|, \ \forall J \in \mathcal{D}\}$$

on the domain $\Omega = \{x \in \mathbb{R}^3 : x_3 \geq 0, |x_1|^p \leq x_3\}$. Note that the Bellman function is a sort of the supremum of the left side of the inequality that we would like to prove (with $MT_\varepsilon(\sum_{k=1}^n d_k)$ replaced with g and $\sum_{k=1}^n d_k$ replaced with f) with all other quantities fixed and $|x_1|^p \leq x_3$ in the domain is just Hölder's inequality. Our goal is to find $\mathcal{B}$. It can be found in [2] that proving (1) is equivalent to showing that $\mathcal{B}(0, 0, x_3) \leq ((p^* - 1)^2 + \tau^2)^{\frac{p}{2}} x_3$. For $p = 2$, $\mathcal{B}(x) = x_2^2 - x_1^2 + (1 + \tau^2)x_3$ is a trivial calculation. However, for $p \neq 2$ it is much more difficult to find $\mathcal{B}$, so we need some properties. We start with simple properties that followed from Minkowski's inequality combined with famous Burkholder's result [2], where he proved (1) for $\tau = 0$: 1) If $2 \leq p < \infty$, $\mathcal{B}(0, 0, x_3) \leq ((p^* - 1)^2 + \tau^2)^{\frac{p}{2}} x_3$; if $1 < p \leq 2$, then $\mathcal{B}(0, 0, x_3) \geq ((p^* - 1)^2 + \tau^2)^{\frac{p}{2}} x_3$. These inequalities are valid for any τ. However, the converse inequalities seem to be far from being simple. We can prove them only for relatively small τ (see the explanation in Section 5), and we know that the best constant $((p^* - 1)^2 + \tau^2)^{\frac{1}{2}}$ in (1) changes its form if $|\tau|$ becomes large ("phase transition") in the case $1 < p < 2$.

2. Bellman function properties

Since $\mathcal{B}$ is independent of the initial choice of I (of finite length) and $\{\alpha_{m,n}\}_{m,n}$, we will return them to being arbitrary.

Lemma 2.1 *(Weak Concavity) Suppose $x^\pm \in \Omega$ such that $x = \alpha^+ x^+ + \alpha^- x^-$, $\alpha^+ + \alpha^- = 1$. If $|x_1^+ - x_1^-| = |x_2^+ - x_2^-|$ then $\mathcal{B}(x) \geq \alpha^+\mathcal{B}(x^+) + \alpha^-\mathcal{B}(x^-)$.*

Observe that $|x_1^+ - x_1^-| = |x_2^+ - x_2^-|$ is just the condition $|(f, h_J)| = |(g, h_J)|$. This condition can be made more manageable but making the following change of coordinates: $y_1 := \frac{x_2+x_1}{2}$, $y_2 := \frac{x_2-x_1}{2}$ and $y_3 := x_3$. Also, let $\mathcal{M}(y_1, y_2, y_3) := \mathcal{B}(x_1, x_2, x_3) = \mathcal{B}(y_1 - y_2, y_1 + y_2, y_3)$ and $\Xi := \{y \in \mathbb{R}^3 : y_3 \geq 0, |y_1 - y_2|^p \leq y_3\}$ denote the Bellman function and domain in the y-variable. Note that $|x_1^+ - x_1^-| = |x_2^+ - x_2^-|$ is equivalent to either y_1 or y_2 being fixed. Therefore an equivalent way to check weak concavity is the following.

Lemma 2.2 *Fix $j \neq i \in \{1, 2\}$ and y_i as $y_i^+ = y_i^-$. Then $\mathcal{M}_{y_j y_j} \leq 0, \mathcal{M}_{y_3 y_3} \leq 0$ and $D_j = \mathcal{M}_{y_j y_j}\mathcal{M}_{y_3 y_3} - (\mathcal{M}_{y_j y_3})^2 \geq 0$ is equivalent to $\mathcal{M}$ being a concave function of y_j, y_3.*

The remaining Bellman function properties can now be easily derived.

Proposition 2.1 *Suppose $\mathcal{M}$ is $C^1(\mathbb{R}^3)$, then $\mathcal{M}$ has the following properties: (i) Symmetry: $\mathcal{M}(y_1, y_2, y_3) = \mathcal{M}(y_2, y_1, y_3) = \mathcal{M}(-y_1, -y_2, y_3)$; (ii) Dirichlet boundary data: $\mathcal{M}(y_1, y_2, (y_1 - y_2)^p) = ((y_1 + y_2)^2 + \tau^2(y_1 - y_2)^2)^{\frac{p}{2}}$; (iii) Neumann conditions: $\mathcal{M}_{y_1} = \mathcal{M}_{y_2}$ on $y_1 = y_2$ and $\mathcal{M}_{y_1} = -\mathcal{M}_{y_2}$ on $y_1 = -y_2$; (iv) Homogeneity: $\mathcal{M}(ry_1, ry_2, r^p y_3) = r^p\mathcal{M}(y_1, y_2, y_3), \forall r > 0$; (v) Homogeneity relation: $y_1\mathcal{M}_{y_1} + y_2\mathcal{M}_{y_2} + py_3\mathcal{M}_{y_3} = p\mathcal{M}$.*

An added assumption simplifies the problem

Lemma 2.2 introduces a partial differential inequality into our problem: $\mathcal{M}_{y_j y_j}\mathcal{M}_{y_3 y_3} - (\mathcal{M}_{y_j y_3})^2 \geq 0$. Rather than dealing with this difficulty, we add an assumption.

Assumption 1 Fix $i \neq j \in \{1, 2\}$ and y_i. Then the 2×2 Hessian matrix $\{\mathcal{M}_{y_k y_\ell}\}_{k,\ell=j,3}$ is degenerate.

The degeneracy of the above matrix gives us the well known Monge–Ampère Equation equation, $\mathcal{M}_{y_j y_j}\mathcal{M}_{y_3 y_3} - (\mathcal{M}_{y_j y_3})^2 = 0$, which has a solution.

Theorem 2.3 *(Pogorelov 1956)For $j = 1$ or 2, $\mathcal{M}_{y_j y_j}\mathcal{M}_{y_3 y_3} - (\mathcal{M}_{y_3 y_j})^2 = 0$ has the solution $M(y) = y_j t_j + y_3 t_3 + t_0$ on the characteristics $y_j dt_j + y_3 dt_3 + dt_0 = 0$, which are straight lines in the $y_j \times y_3$ plane.*

Furthermore, t_0, t_j, t_3 are constant on characteristics with the property $M_{y_j} = t_j, M_{y_3} = t_3$.

3. Bellman function candidate

The characteristics for Monge–Ampère solution, in Theorem 2.3 can only behave one of four possible ways in the $y_j \times y_3$ plane. Using the Bellman function properties shown in Proposition 2.1 and checking that the weak concavity in Lemma 2.2 is satisfied we can deduce the following Bellman function candidate. Refer to [7] and [8] for more details of this method.

Proposition 3.1 *Denote $G(z_1, z_2) = (z_1 + z_2)^{p-1}[z_1 - (p-1)z_2], \omega = \left(\frac{M(y)}{y_3}\right)^{\frac{1}{p}}$ and $\gamma = \frac{1-\tau^2}{1+\tau^2}$. Then, for $|\tau| \le \frac{1}{2}$, the Monge–Ampère solution that satisfies all Bellman function properties is the following.*

***(A)** For $p \in (1, 2], M(y) = (1+\tau^2)^{\frac{p}{2}}[y_1^2+2\gamma y_1 y_2+y_2^2]^{\frac{p}{2}}+\left(\frac{1}{(p-1)^2}+\tau^2\right)^{\frac{p}{2}}[y_3-(y_1-y_2)^p], y_2 \in [\frac{2-p}{p}y_1, y_1)$ and M is given by implicit equation $G(y_1 - y_2, y_1 + y_2) = y_3 G(1, \sqrt{\omega^2 - \tau^2}), y_2 \in (-y_1, \frac{2-p}{p}y_1]$.*

***(B)** For $p \in [2, \infty), M(y) = (1 + \tau^2)^{\frac{p}{2}}[y_1^2 + 2\gamma y_1 y_2 + y_2^2]^{\frac{p}{2}} + ((p-1)^2 + \tau^2)^{\frac{p}{2}}[y_3 - (y_1 - y_2)^p], y_2 \in (-y_1, \frac{p-2}{p}y_1]$ and M is given implicitly by $G(y_1 + y_2, y_1 - y_2) = y_3 G(\sqrt{\omega^2 - \tau^2}, 1), y_2 \in [\frac{p-2}{p}y_1, y_1)$.*

Remarks. 1) For $p \in (1, 2)$ one gets easily $M(\frac{x_1+x_2}{2}, \frac{x_2-x_1}{2}, x_3) \le \mathcal{B}(x_1, x_2, x_3)$; 2) If $p \in [2, \infty)$ one gets easily $\mathcal{B}(x_1, x_2, x_3) \le M(\frac{x_1+x_2}{2}, \frac{x_2-x_1}{2}, x_3)$. Both relationships are easy consequences of Minkowski's inequality. Both of them are true for any τ. We will see that for relatively small τ the converse inequalities hold (and so we have equalities). This breaks down for large $|\tau|$. 3) Equality $\mathcal{B} = M(\frac{x_1+x_2}{2}, \frac{x_2-x_1}{2}, x_3)$ obtained in the next section obviously implies the calculation of the best constant in $\mathcal{B}(0, 0, x_3) \le c(\tau, p)x_3$, it turns out that $c(\tau, p) = ((p^*-1)^2 + \tau^2)^{\frac{p}{2}}$, which gives (1).

4. Our Bellman function candidate is actually the Bellman function

Since $B, \mathcal{B}, M, \mathcal{M}$ actually depend on τ, we will use $B_\tau, \mathcal{B}_\tau, M_\tau, \mathcal{M}_\tau$. We will now revert to x rather than y for the variable, Ω rather than Ξ for the domain and $\mathcal{B}$ rather than $\mathcal{M}$ for the the Bellman function. In Proposition 3.1 we obtained a Bellman function candidate that we think is the Bellman function. Let $B(x_1, x_2, x_3) = M(\frac{x_1+x_2}{2}, \frac{x_2-x_1}{2}, x_3)$ denote the Bellman function candidate from Proposition 3.1. We must now show that $B = \mathcal{B}$. This is the price that we must pay for trading the partial differential inequality for the Monge–Ampère equation.

Proposition 4.1 *If $1 < p \le 2$ and $|\tau| \le 1/2$ then B_τ is bi-concave in Ω, that is B_τ is concave in $(x_1 + x_2, x_3)$ and in $(x_2 - x_1, x_3)$.*

Proposition 4.2 *If some function $\widetilde{B}$ is bi-concave with $\widetilde{B}(x, y, |x|^p)) \ge (\tau^2 x^2 + y^2)^{\frac{p}{2}}$, then $\widetilde{B} \ge \mathcal{B}_\tau$. In particular, $B_\tau \ge \mathcal{B}_\tau$, $|\tau| \le 1/2$, $1 < p \le 2$. Also $B_\tau \ge \mathcal{B}_\tau$ for all $p \in [2, \infty)$ just by Minkowski's inequality.*
This follows in exactly the same way as in [8]. Notice that for $2 \le p < \infty$ the inequality $B_\tau \ge \mathcal{B}_\tau$ is true for any τ (this can be proved just by Minkowski's inequality).

Proposition 4.3 *For $1 < p < \infty, B_\tau \le \mathcal{B}_\tau$.*
We already mentioned that for $1 < p \le 2$ the use of Minkowski's inequality will show this result for any τ. For $2 \le p < \infty$, the use of the same extremal function sequence as in [7] will give the result.

5. Main result

Let $U_{p,\tau}(x,y) = \begin{cases} V_{p,\tau}(x,y) := (\tau^2|x|^2 + |y|^2)^{\frac{p}{2}} - ((p^*-1)^2+\tau^2)^{\frac{p}{2}}|x|^p & : |y| \le (p^*-1)|x| \\ p\left(1-\dfrac{1}{p^*}\right)^{p-1}\left(1+\dfrac{\tau^2}{(p^*-1)^2}\right)^{\frac{p-2}{2}}(|x|+|y|)^{p-1}[|y|-(p^*-1)|x|] & : |y| \ge (p^*-1)|x| \end{cases}$
and $2 < p < \infty$. Then the Bellman function in Proposition 3.1 can be more compactly written as, the unique positive solution to

$$U_{p,\tau}\left(x_1, x_2\right) = U_{p,\tau}\left(x_3^{\frac{1}{p}}, \sqrt{B_\tau^{\frac{2}{p}} - \tau^2 x_3^{\frac{2}{p}}}\right). \tag{2}$$

One can show that $U_{p,\tau}$ is the least biconcave majorant of $V_{p,\tau}$. Similarly for $1 < p \le 2, U_{p,\tau}$ (with the two pieces of the above function interchanged) is the least biconcave majorant of $V_{p,\tau}$ and the Bellman function satisfies the relation in (2). This allows us to prove the next theorem. But first let us explain the role of smallness of τ. Comparing B_τ with Burkholder's function $B = B_0$ we see that $B_\tau^{2/p} = B^{2/p} + \tau^2 x_3^{2/p}$. We wish B_τ to be bi-concave in (x_1+x_2, x_3) and (x_2-x_1, x_3), B has these properties. Let us think that $2/p > 1$, take the second derivative of B_τ in x_3. When τ is small, it is easy to see that this second derivative is still negative (as this is so for B). But if τ is not small the convexity of $\tau^2 x_3^{2/p}$ may (and will) start to win over the concavity of B.
Theorem 5.1 *Let $\tau \in [-\frac{1}{2}, \frac{1}{2}], I = [0,1], \langle f\rangle_I = x_1, \langle g\rangle_I = x_2, g$ be the martingale transform of f, in terms of the Haar expansion, and $|x_2| \le |x_1|$. Then $\langle(\tau^2|f|^2+|g|^2)^{\frac{p}{2}}\rangle_I \le ((p^*-1)^2+\tau^2)^{\frac{p}{2}}\langle|f|^p\rangle_I$ with sharp constant, where $p^*-1 = \max\{p-1, \frac{1}{p-1}\}$. For $p \in [2,\infty)$ the estimate from above is true for any τ. For $p \in (1,2)$ the estimate from above is no longer true for large τ. For $p \in (1,\infty)$ the sharpness of the constant (the estimate from below) is true for any τ.*
We have the application for the Riesz transforms. The lower bound is a generalization of the lower bound technique in [4] to vector-valued multipliers. The upper bound is a generalization of the result in [1] to the vector setting.
Theorem 5.2 *Let $R_i, i = 1,2$ be the Riesz transform on the plane and $p \in (1,\infty)$. Then for $|\tau| \le 1/2$ we can calculate the norm in $L^p(\mathbb{C})$:*

$$\left\|\left(\, R_1^2 - R_2^2\,,\, \tau I\,\right)\right\|_{L^p(\mathbb{C},\mathbb{R})\to L^p(\mathbb{C},\mathbb{R}^2)} = ((p^*-1)^2+\tau^2)^{\frac{1}{2}}\,,$$

where the estimate from above for the norm is true even for all τ if $p \in [2,\infty)$.
Remark. The estimate in this theorem is on real valued functions. We expect that the same is true for complex valued functions.

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